Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Geometry and Physics 51 (2004) 269–331
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

Review

Fredholm–Lagrangian–Grassmannian and the Maslov index

Kenro Furutani*

Department of Mathematics, Faculty of Science and Technology, Science University of Tokyo, 2641 Noda, Chiba 278-8510, Japan

Received 4 November 2003; received in revised form 25 March 2004; accepted 3 April 2004

Abstract

This is a review article on the topology of the space, so called, Fredholm–Lagrangian–Grassmannian and the quantity "*Maslov index*" for paths in this space based on the standard theory of functional analysis. Our standing point is to define the Maslov index for arbitrary paths in terms of the fundamental spectral property of the Fredholm operators as an intersection number with the "Maslov cycle". This argument was first recognized by J. Phillips and was used to define the "*Spectral flow*" not only for loops but also for arbitrary paths of selfadjoint Fredholm operators. We make the arguments as elementary as possible.

MSC: 53D12; 58J30; 47A53; 58J50

JGP SC: Global differential geometry; Global analysis

Keywords: Symplectic Hilbert space; Lagrangian subspace; Fredholm pair; Maslov index; Leray index; Kashiwara index; Hörmander index; Fredholm operator; Fredholm–Lagrangian–Grassmannian; Elliptic operator; K-group; Unitary group

1. Introduction

This is a review article. We develop a unified theory of the topology of the space "*Fredholm–Lagrangian–Grassmannian*" and the theory of the "Maslov index" for arbitrary paths in this space. Most of the contents in this article are treated in the papers [4–6,14].

* Tel.: +81-471241501; fax: +81-471239762.
E-mail address: furutani_kenro@ma.noda.tus.ac.jp (K. Furutani).

0393-0440/$ – see front matter
doi:10.1016/j.geomphys.2004.04.001

Also there are already many papers which treat with more or less similar topics with this paper ([8–11,16,24,29–31] and others).

Even so our method to treat with this topics, especially the treatment of the Maslov index is different from other papers and so, the whole theory should be rewritten in a self-contained and complete form for being well understood and we hope this article would provide a reasonable framework of this subject. We would like to emphasis here that the method for defining the Maslov index for paths with fixed end points is quite natural and elementary as an intersection number with the "Maslov cycle". This follows from the basic spectral property of the Fredholm operators and the method is of course valid for finite dimensional cases. We believe that these points must be important, and should be known widely, since in the applications it naturally appears the requirement to treat with such an integer for not only loops, but also paths of Lagrangian subspaces in an intrinsic way. Here neither we need any generic arguments which was assumed in the paper [29], nor we rely on quantities, "*Leray index*" and "*Kashiwara index*" which are only defined for the finite dimensional cases [8,16,17,24,30].

There are many places in which the Maslov index and related quantities appear, and so here we do not mention them, since they are explained and treated in many articles cited above according to their subjects. Here we only concentrate to explain the basic theory of the topology of the Fredholm–Lagrangian–Grassmannian and the Maslov index for paths from the point of view of the standard theory of functional analysis.

The main method in this article is in the analysis of operators on Hilbert spaces but the arguments should be carefully carried out, simply because it is in the infinite dimension. There are many parts which are similar to finite dimensional cases, but also there are many parts which are not just a generalization of the finite dimensional cases. We will make clear the differences of the infinite dimensional case from finite dimensional cases.

We avoid to base on a general theory of the infinite dimensional manifold theory and try the treatments as elementally as possible and to be self-contained. However we must recognize several highly non-trivial facts like:

(a) Kuiper's Theorem A.1.
(b) Palais's Theorem A.3.
(c) The spaces of certain class of Fredholm operators are identified as classifying spaces for K and KO-groups.

In Section 2 we just begin from the basic facts in symplectic Hilbert space and the space of their Lagrangian subspaces. Especially we explain the "*Souriau map*" precisely and give a proof for determining the fundamental group of the Fredholm–Lagrangian–Grass- mannian. In Section 3 we define the Maslov index for paths and the "Hörmander index" in the infinite dimension and construct the universal covering space of the Fredholm–Lagrangian–Grassmannian. Also we discuss the Maslov index with the relations between certain bilinear forms. In Section 4 we summarize the finite dimensional cases and extend the quantity "Kashiwara index" ("cross index") to any triples of unitary operators. In Section 5 we treat with polarized symplectic Hilbert spaces and prove a symplectic reduction theorem in the infinite dimension. Finally in Section 6 we explain an example in this framework and a formula relating with "*Spectral Flow*".

2. Symplectic Hilbert space and Lagrangian subspace

We start from the definition of the symplectic Hilbert spaces and their isotropic, involutive and Lagrangian subspaces and operations among them.

2.1. Symplectic Hilbert space

Let $(H, \langle \bullet, \bullet \rangle, \omega)$ be a (real and separable) Hilbert space with an inner product $\langle \bullet, \bullet \rangle$ and we assume H has a symplectic form $\omega(\bullet, \bullet)$, that is, a non-degenerate, skew-symmetric bounded bilinear form.

Here we mean that the bilinear form ω is non-degenerate in such a sense that the linear map

$$\omega^{\#} : H \to H^* \quad (= \text{dual space}), \qquad \omega^{\#}(x)(y) = \omega(x, y) \tag{2.1}$$

is an isomorphism between the Hilbert space H and its dual space H^*. In finite dimensional cases, the injectivity of the map $\omega^{\#}$ implies that it is an isomorphism, but in the infinite dimension this does not hold automatically. In our case we call the Hilbert space as a symplectic Hilbert space.

In the theory below we do not replace the symplectic form ω after once it is introduced on the real Hilbert space H, but we may always assume that there exists an orthogonal transformation $J : H \to H$ such that $\omega(x, y) = \langle Jx, y \rangle$ for any $x, y \in H$ and $J^2 = -\mathrm{Id}$ by replacing the inner product with another one which defines an equivalent norm on H.

We give the proof of this fact in Appendix D.

So from the beginning we can assume the following relations:

$${}^tJ = -J, \quad \langle Jx, Jy \rangle = \langle x, y \rangle \quad \text{and}\, \omega(Jx, Jy) = \omega(x, y) \quad \text{for all}\, x, y \in H.$$

Here ${}^tJ$ denotes the transpose of J with respect to the inner product $\langle \bullet, \bullet \rangle$. In this case we call these quantities, the symplectic form ω, the inner product $\langle \bullet, \bullet \rangle$ and the almost complex structure J are compatible each other.

Example 2.1. Let E be a real separable Hilbert space and E^* its dual space. We denote the identification between E and E^*, by $\mathcal{D} : E \to E^*$; $E \ni x \mapsto \mathcal{D}(x)(\bullet) = \langle \bullet, x \rangle \in E^*$ (Riesz Representation Theorem). We can introduce an inner product on the dual space E^* through the map $\mathcal{D}$ in an obvious way and then the direct sum $H = E \oplus E^*$ has a naturally defined skew-symmetric bilinear form

$$\omega : H \times H \to \mathbb{R},$$

$$\omega(x \oplus \phi, y \oplus \psi) = \psi(x) - \phi(y) = \langle J(x \oplus \phi), y \oplus \psi \rangle,$$

where the almost complex structure $J : H \to H$ is given as

$$J(x \oplus \phi) = \mathcal{D}^{-1}(\phi) \oplus -\mathcal{D}(x).$$

Example 2.2. Let A be a densely defined closed symmetric operator on a Hilbert space L. Let $\mathfrak{D}(A)$ (respectively $\mathfrak{D}(A^*)$) be the domain of A (respectively A^*) and we impose the graph inner product on $\mathfrak{D}(A^*)$: $\langle x, y\rangle^{\mathcal{G}} = \langle x, y\rangle + \langle A^*(x), A^*(y)\rangle$. Then $\mathfrak{D}(A^*)$ becomes a Hilbert space and $\mathfrak{D}(A)$ is a closed subspace in $\mathfrak{D}(A^*)$ with respect to this graph norm. Let $\boldsymbol{\beta}$ be the factor space $\boldsymbol{\beta} = \mathfrak{D}(A^*)/\mathfrak{D}(A)$. We can introduce a non-degenerate anti-symmetric bilinear form ω on $\boldsymbol{\beta}$ by

$$\omega([x], [y]) = \langle A^*(x), y\rangle - \langle x, A^*(y)\rangle, \tag{2.2}$$

where we denote by $[x]$, the class of $x \in \mathfrak{D}(A^*)$ in $\boldsymbol{\beta}$.

It will be apparent of the well-definedness of the form ω just from the definition of the adjoint operator.

We will note the non-degeneracy of the form ω: the factor space $\boldsymbol{\beta}$ is identified with the orthogonal complement $\mathfrak{D}(A)^{\perp}$ of $\mathfrak{D}(A)$ in $\mathfrak{D}(A^*)$ with respect to the graph inner product. It is characterized as follows:

$$\mathfrak{D}(A)^{\perp} = \{x \in \mathfrak{D}(A^*) | A^*(x) \in \mathfrak{D}(A^*) \ \text{ and } \ A^*(A^*(x)) = -x\}.$$

From this characterization we know at once that A^* restricted to $\mathfrak{D}(A)^{\perp}$ is an orthogonal transformation into itself and defines an almost complex structure on $\mathfrak{D}(A)^{\perp}$ and moreover we have

$$\begin{aligned}\omega([x], [y]) &= \langle A^*(x), y\rangle - \langle x, A^*(y)\rangle = \langle A^*(x), y\rangle + \langle A^*(A^*(x)), A^*(y)\rangle \\ &= \langle A^*(x), y\rangle^{\mathcal{G}}.\end{aligned}$$

This equality shows that our Hilbert space $\boldsymbol{\beta}$ with the symplectic form ω defined above together with the almost complex structure A^* (after being identified with the orthogonal complement $\mathfrak{D}(A)^{\perp}$) is a symplectic Hilbert space with a compatible symplectic form, inner product and the almost complex structure.

We will deal with this example in Section 6 together with a homotopy invariant, so called, "*Spectral flow*" of a family of selfadjoint Fredholm operators.

Example 2.3. Let $\pi : E \to M$ be a real vector bundle on a manifold M with a bundle map of almost complex structures $J : E \to E$, $J^2 = -\mathrm{Id}$. By introducing a suitable inner product on E and a (smooth) measure on M we have a Hilbert space $L_2(M, E)$ of L_2-sections of E with a symplectic form defined by the bundle map J in an obvious way.

When we regard the real Hilbert space H as a complex Hilbert space through the almost complex structure J with the Hermitian inner product $\langle \bullet, \bullet\rangle_J = \langle \bullet, \bullet\rangle - \sqrt{-1}\omega(\bullet, \bullet)$, we denote it by H_J, and we denote the group of unitary transformations on H_J by

$$\mathcal{U}(H_J) = \{U \in \mathcal{B}(H) | UJ = JU \ \text{ and } \ {}^tUU = U\,{}^tU = \mathrm{Id}\},$$

where $\mathcal{B}(H)$ denotes the space of bounded linear operators on the real Hilbert space H.

For a subspace μ in H, let us denote by μ° the annihilator of μ with respect to ω:

$$\mu^{\circ} = \{x \in H | \omega(x, y) = 0 \ \text{ for all } y \in \mu\}, \tag{2.3}$$

and we denote the orthogonal complement (with respect to the fixed inner product $\langle \bullet, \bullet\rangle$ on H) of μ by $\mu^{\perp}$.

Note that we know easily by the definition that for any subspace μ the annihilator μ° is closed by the similar way to prove the closedness of the orthogonal compliment $\mu^\perp$. Also by the non-degeneracy assumption of the symplectic form, we have the idempotentness of the operation $\mu \mapsto \mu^\circ$. In general we have

Proposition 2.4. $(\mu^\circ)^\circ = \bar{\mu}$.

Proof. By the definition of the annihilator it will be apparent that $\bar{\mu} \subset (\mu^\circ)^\circ$. Let $z_0 \in (\mu^\circ)^\circ$ and assume that $z_0 \notin \bar{\mu}$, then there is a bounded linear functional f on H such that $f = 0$ on μ and $f(z_0) \neq 0$. By the non-degeneracy assumption of the symplectic form ω, we have an element $u_0 \in H$ such that $f(x) = \omega(x, u_0)$. Then $u_0 \in \mu^\circ$, but $\omega(z_0, u_0) \neq 0$. This is a contradiction. So there are no such z_0. □

The following properties will be proved easily.

Proposition 2.5. *Let μ, ν be subspaces in H, then*

$$(\mu + \nu)^\circ = \mu^\circ \bigcap \nu^\circ, \tag{2.4}$$

$$\left(\mu \bigcap \nu\right)^\circ = \overline{\mu^\circ + \nu^\circ}. \tag{2.5}$$

As in the same way with finite dimensional cases we characterize a subspace $\mu \in H$ in the following definition.

Definition 2.6.

(a) Isotropic, if $\mu \subset \mu^\circ$.
(b) Lagrangian, if $\mu^\circ = \mu$.
(c) Coisotropic (or involutive), if $\mu^\circ \subset \mu$.
(d) Symplectic, if μ is closed and $\mu + \mu^\circ = H$ (= direct sum).

By the compatibility assumption among the symplectic form ω, the inner product $\langle \bullet, \bullet \rangle$ and the almost complex structure J the following properties hold.

Proposition 2.7.

(a) *If μ is isotropic, then $J(\mu)$ is also isotropic and $\mu \perp J(\mu)$.*
(b) *If μ is Lagrangian, then μ is a closed subspace, $J(\mu)$ is also Lagrangian and $J(\mu) = \mu^\perp$. Conversely let μ be a closed subspace and assume that $\mu^\perp = J(\mu)$, then μ is a Lagrangian subspace.*
(c) *If μ is coisotropic, then $J(\mu)$ is also coisotropic.*

If μ is symplectic, then $\mu + \mu^\circ$ is a direct sum, however it is not always orthogonal.

Proposition 2.8. *If μ is symplectic, then the restriction of the map $\omega^\#$ to each of μ and μ° is isomorphic with μ^* and $(\mu^\circ)^*$, respectively. So, by replacing the inner product with* *a suitable one so that we can assume that μ and μ° are orthogonal and then each is a symplectic Hilbert space with the compatible structure.*

Proof. If we embed μ^* into H^* by extending $f \in \mu^*$ to $\tilde{f}$ being zero on μ°, then for any f there is an element $a+b \in \mu + \mu^\circ$ such that $\omega^\#(a+b) = \tilde{f}$ and from the assumption, b must be zero, that is, we have $(\omega_{|\mu\times\mu})^\# = (\omega^\#)_{|\mu}$. Hence μ is a symplectic Hilbert space. So is μ°. Then the rests of the proposition will follow easily from Proposition D.1. □

Remark 2.9.

(a) Let E be a finite dimensional subspace in H such that $E \bigcap E^\circ = \{0\}$, then E is symplectic in the above sense of Definition 2.6(d), that is $E \oplus E^\circ = H$.
(b) Let λ be a Lagrangian subspace and L be a closed subspace in λ. Put $H_1 = L + J(L)$ and $H_2 = L^\perp \bigcap \lambda + J(L^\perp \bigcap \lambda)$, then H_1 and H_2 are symplectic, of course with the compatibility assumption of the symplectic structure on H.

2.2. *Lagrangian–Grassmannian*

Let $\Lambda(H)$ denote the space of all Lagrangian subspaces of H. We call this space *Lagrangian–Grassmannian* of the symplectic Hilbert space H.

Remark 2.10.

(a) Let $\lambda \in \Lambda(H)$, then by the above Proposition 2.7 we have an orthogonal decomposition $H = \lambda \oplus J(\lambda)$ and by identifying the dual space of λ with $J(\lambda)$ we know that any symplectic Hilbert space is isomorphic with the Example 2.1.
(b) In the symplectic Hilbert space a maximum isotropic subspace is always a Lagrangian subspace. For the symplectic Banach space (this is defined by the same way as symplectic Hilbert spaces) a maximal isotropic subspace need not be a Lagrangian subspace, moreover there is a symplectic Banach space which has no Lagrangian subspace (see [23]). This fact says that a symplectic Banach space is not necessarily isomorphic with a standard one of the form $V \oplus V^*$ with a reflexive Banach space V. In this article we do not treat with the symplectic Banach space.

We denote by $\mathcal{P}_\lambda$ the orthogonal projection operator in H onto the subspace λ. With this correspondence we embed $\Lambda(H)$ into $\mathcal{B}(H)$ as a closed subset (see Corollary 2.12 below for the closedness):

$$\mathcal{P} : \Lambda(H) \to \mathcal{B}(H), \qquad \lambda \mapsto \mathcal{P}_\lambda. \tag{2.6}$$

Then it will be natural to introduce the metric d on the space $\Lambda(H)$ as the difference of the norms of the corresponding projection operators: $d(\lambda,\mu) = \|\mathcal{P}_\lambda - \mathcal{P}_\mu\|$. *Henceforth we regard the space $\Lambda(H)$ equipped with this metric always.*

A projection operator in the image of the map $\mathcal{P}$ is characterized by the following proposition.

Proposition 2.11. *Let P be an orthogonal projection operator in H. Then the image $P(H)$ is a Lagrangian subspace, if and only if $J = J \circ P + P \circ J$.*

Proof. Let an orthogonal projection operator P satisfy the relation $J = JP + PJ$, then we have $\omega(P(x), P(y)) = \langle J \circ P(x), P(y)\rangle = \langle J(x) - P \circ J(x), P(y)\rangle = \langle J(x), P(y)\rangle - \langle J(x), P(y)\rangle = 0$. So $P(H)$ is an isotropic subspace. Let assume for any $x \in H$ $\omega(P(x), y) = 0$, then $\langle J(x) - P \circ J(x), y\rangle = 0$. So we have $\langle J(x), y - P(y)\rangle = 0$ for any $x \in H$. Hence $y = P(y)$, and so $P(H)^\circ = P(H)$, that is $P(H)$ is a Lagrangian subspace.

Now assume that $P(H)$ is a Lagrangian subspace. Then we have for $x \in P(H)$, $P \circ J(x) + J \circ P(x) = J \circ P(x) = J(x)$, and for $x \in \mathrm{Ker}(P)$ we have $P \circ J(x) + J \circ P(x) = P \circ J(x) = J(x)$. □

As a corollary of this proposition we have the following corollary.

Corollary 2.12. *The subspace consisting of orthogonal projections whose images are Lagrangian subspaces is closed in the Banach space $\mathcal{B}(H)$.*

The group $\mathcal{U}(H_J)$ acts on $\Lambda(H)$ in an obvious way.

Proposition 2.13. *The action $\mathcal{U}(H_J) \times \Lambda(H) \to \Lambda(H)$ is continuous.*

Proof. From the relation

$$\mathcal{P}_{U(\mu)} = U \circ \mathcal{P}_\mu \circ U^{-1}, \tag{2.7}$$

we have

$$\begin{aligned}\mathcal{P}_{U(\mu)} - \mathcal{P}_{V(\nu)} &= U \circ \mathcal{P}_\mu \circ U^{-1} - V \circ \mathcal{P}_\mu \circ V^{-1} \\ &= U \circ (\mathcal{P}_\mu - \mathcal{P}_\nu) \circ U^{-1} + (U - V) \circ \mathcal{P}_\nu \circ U^{-1} \\ &\quad + V \circ \mathcal{P}_\nu \circ (U^{-1} - V^{-1}),\end{aligned}$$

and this formula shows the continuity of the action. □

By fixing an $\ell \in \Lambda(H)$ we have a surjective map ρ_ℓ:

$$\rho_\ell : \mathcal{U}(H_J) \to \Lambda(H), \qquad U \mapsto U(\ell^\perp). \tag{2.8}$$

Theorem 2.14. *The map (2.8) defines a principal fiber bundle with the structure group $\mathcal{O}(\ell)$, the group of orthogonal transformations on ℓ, and by Kuiper's Theorem A.1 it is a trivial bundle and the space $\Lambda(H)$ itself is also contractible.*

Remark 2.15. Of course the triviality of this bundle is not true for the finite dimensional case.

Corollary 2.16. *The map ρ_ℓ is an open map and the topology on $\Lambda(H)$ coincides with the quotient topology of $\mathcal{U}(H_J)$ by the map ρ_ℓ.*

Theorem 2.14 is proved if we have local sections of the map ρ_ℓ. Here we construct local sections in two ways. Because both of the arguments contain several interesting properties of the space $\Lambda(H)$ and relating properties of projection operators.

2.2.1. First method

We begin from a lemma.

Lemma 2.17. *Let P and Q be two projection operators on the Hilbert space H, and assume that* $\|P-Q\|<1$. *Put*

(a) $A=(1-P)(1-Q)+PQ$,
(b) $B=(1-Q)(1-P)+QP$,
(c) $C=1-(P-Q)^2$,
(d) $D=\sum_{n=0}^{\infty}\alpha_n(P-Q)^{2n}$, *where* $(1-x)^{-1/2}=\sum_{n=0}^{\infty}\alpha_n x^n$ *is the Taylor expansion.*

Then we have

(a) $AB=BA=C$,
(b) $D^2C=CD^2=I$,
(c) $P(P-Q)^2=(P-Q)^2P$, $Q(P=Q)^2=(P-Q)^2Q$,
(d) $DP=PD$, $DQ=QD$.

Proof. All these will be proved by direct calculations. Note that all of the operators A, B, C and D are, as a result, invertible and $AC=CA$, $CB=BC$ and $DC=CD$. □

Now put $W=DA$, then we have the following proposition.

Proposition 2.18.

(a) *W is invertible and the inverse is given by* $W^{-1}=BD$,
(b) $WQ=PW$.

Hence we have $W(Q(H))=P(W(H))$. *Moreover if both of P and Q are orthogonal projections, the operator W is unitary, that is the ranges of the projections P and Q are transformed each other by a unitary operator W.*

Proof. $WQ=D((1-P)(1-Q)+PQ)Q=DPQ$ and $PW=PDA=DPA=DP((1-P)(1-Q)+PQ)=DPQ$. Since $(DA)(BD)=DCD=1$ and $(BD)(DA)=BD^2A=BC^{-1}A=1$ the operator W is invertible. Also since $W^*=A^*D^*$, if both of P and Q are orthogonal we have $W^*=BD=W^{-1}$, that is, W is unitary and give a unitary equivalence of the projections P and Q. □

Let $\mu\in\Lambda(H)$ and $\mathbf{V}_\mu=\{\nu|\|\mathcal{P}_\nu-\mathcal{P}_\mu\|<1\}$, an open neighborhood of μ, where $\mathcal{P}_\nu$ denote the orthogonal projection operator with the image ν.

Now we describe a local section $s_\mu^{(1)}:\mathbf{V}_\mu\to\mathcal{U}(H_J)$ of the map $\rho_\ell:\mathcal{U}(H_J)\to\Lambda(H)$. For this purpose we fix a unitary operator V_0 such that $V_0(\ell^\perp)=\mu$ and define

$$s_\mu^{(1)}:\mathbf{V}_\mu\ni\nu\to W_\nu^{-1}\circ V_0, \tag{2.9}$$

where we denote $W_\nu = (1-(\mathcal{P}_\mu - \mathcal{P}_\nu)^2)^{-1/2}((1-\mathcal{P}_\mu)(1-\mathcal{P}_\nu) + \mathcal{P}_\mu\mathcal{P}_\nu)$. The continuity of this section will be apparent from the expression.

2.2.2. Second method

Let $\lambda \in \Lambda(H)$.

Notation 2.19. $\mathbf{O}_\lambda = \{\mu \in \Lambda(H) | \mu \text{ is transversal to } \lambda\}$. Note that we mean "*transversal*" by the condition: $\lambda + \mu = H$.

The subset $\mathbf{O}_{\lambda^\perp}$ is an open neighborhood of λ. We denote by $\hat{\mathcal{B}}(\lambda)$ the space of selfadjoint bounded operators on the real Hilbert space λ. Then we have a bijection

$$G_\lambda : \hat{\mathcal{B}}(\lambda) \to \mathbf{O}_{\lambda^\perp}$$

defined by

$$G_\lambda : \hat{\mathcal{B}}(\lambda) \ni A \to G_\lambda(A) = \{x + JA(x) | x \in \lambda\} \in \mathbf{O}_{\lambda^\perp}.$$

By the identification $H_J \cong \lambda \otimes \mathbb{C}$ we regard $A \in \hat{\mathcal{B}}(\lambda)$ as a selfadjoint operator on the complex Hilbert space H_J. Let $A = \int_{-\infty}^{\infty} t \,\mathrm{d}E_t(A)$ be the spectral decomposition of the selfadjoint operator A with the spectral measure $\{E_t(A)\}_{t\in\mathbb{R}}$. We define a unitary operator U_A by

$$U_A = \int_{-\infty}^{\infty} \sqrt{\frac{1+\sqrt{-1}t}{1-\sqrt{-1}t}}\,\mathrm{d}E_t(A),$$

then

$$\int_{-\infty}^{\infty} (1+\sqrt{-1}t)\,\mathrm{d}E_t(A) = U_A \circ \int_{-\infty}^{\infty} \sqrt{1+t^2}\,\mathrm{d}E_t(A). \tag{2.10}$$

Since $\int_{-\infty}^{\infty} \sqrt{1+t^2}\,\mathrm{d}E_t(A)(\lambda) = (\mathrm{Id} + A^2)^{1/2}(\lambda) = \lambda$, we have

$$U_A(\lambda) = \int_{-\infty}^{\infty} (1+\sqrt{-1}t)\,\mathrm{d}E_t(A)(\lambda) = G_\lambda(A). \tag{2.11}$$

Note that $U_A^2 = (\sqrt{-1}\,\mathrm{Id} - A)(\sqrt{-1}\,\mathrm{Id} + A)^{-1}$ is the Cayley transformation of the operator A.

Now fix a unitary operator V such that $V(\ell^\perp) = \lambda^\perp$, then the correspondence

$$s^{(2)}_{\lambda^\perp} : \mathbf{O}_{\lambda^\perp} \ni \mu \to U_A \circ J \circ V, \tag{2.12}$$

gives a local section of the map

$$\rho_\ell : \mathcal{U}(H_J) \to \Lambda(H), \qquad U \mapsto U(\ell^\perp).$$

We must show the continuity of this section $s^{(2)}_{\lambda^\perp}$. This is proved by showing two facts:

(a) the continuity of the correspondence

$$\hat{\mathcal{B}}(\lambda) \ni A \mapsto U_A = \int_{-\infty}^{\infty} \sqrt{\frac{1+\sqrt{-1}t}{1-\sqrt{-1}t}}\, \mathrm{d}E_t(A) \in \mathcal{U}(H_J)$$

with respect to the norm topology and,

(b) the map G_λ is an isomorphism between the spaces $\hat{\mathcal{B}}(\lambda)$ and $\mathbf{O}_{\lambda^\perp}$.

The first one follows from a more general proposition.

Proposition 2.20 ([3]). *Let H be a Hilbert space (real or complex) and f be a continuous function defined on $\mathbb{R}$, then the map $\hat{\mathcal{B}}(H) \ni A \to f(A) \in \mathcal{B}(H)$ is continuous. Here the operator $f(A) = \int_{-\|A\|}^{\|A\|} f(t)\,\mathrm{d}E_t(A)$ is defined by the spectral decomposition $A = \int_{-\|A\|}^{\|A\|} t\,\mathrm{d}E_t(A)$ of the operator A.*

Proof. Let $\{p_n(t)\}$ $(n = 1, 2, \dots)$ be polynomials which converge uniformly to the continuous function f on an interval $[-N, N]$, then for the operator A whose spectrum $\sigma(A)$ is contained in the open interval $(-N, N)$, the operator $p_n(A) = \sum_{k\geq 0}^{N_n} c_k^n A^k$ is also expressed as

$$p_n(A) = \int_{-N}^{+N} p_n(t)\,\mathrm{d}E_t(A).$$

So we know that $\{p_n(A)\}$ converges to $\int f(t)\,\mathrm{d}E_t(A)$ in the sense of operator norm. The correspondence $A \mapsto p_n(A)$ is apparently continuous on the open subspace $\{A \in \hat{\mathcal{B}}(H) | \sigma(A) \subset (-N, N)\}$ in $\hat{\mathcal{B}}(H)$ and so the map $\hat{\mathcal{B}}(\lambda) \ni A \mapsto f(A) \in \mathcal{B}(\lambda)$ is continuous on each such open subspace $\{A \in \hat{\mathcal{B}}(H) | \sigma(A) \subset (-N, N)\}$. Hence we have the desired result. □

Proposition 2.21. *The map $G_\lambda : A \mapsto G_\lambda(A) = \{x + JA(x) | x \in \lambda\}$ is an isomorphism between the spaces $\hat{\mathcal{B}}(\lambda)$ and $\mathbf{O}_{\lambda^\perp}$. Hence it gives a local chart of $\Lambda(H)$.*

We prove a characterization of an orthogonal projection operator corresponding to a Lagrangian subspace in $\mathbf{O}_{\lambda^\perp}$.

Lemma 2.22. *Let $\mathcal{P}_\mu$ be an orthogonal projection operator onto a Lagrangian subspace $\mu \in \Lambda(H)$. Then $\mu \in \mathbf{O}_{\lambda^\perp}$, if and only if $L_\mu = \mathcal{P}_\mu + 1 - \mathcal{P}_\lambda = \mathcal{P}_\mu + \mathcal{P}_{\lambda^\perp}$ is an isomorphism.*

Proof. If $L_\mu = \mathcal{P}_\mu + \mathcal{P}_{\lambda^\perp}$ is an isomorphism, then since $H = (\mathcal{P}_\mu + \mathcal{P}_{\lambda^\perp})(H) \subset \mu + \lambda^\perp$ we know at once $\mu \in \mathbf{O}_{\lambda^\perp}$.

Conversely let us assume μ and $\lambda^\perp$ are transversal. Then there is a bounded operator $A \in \hat{\mathcal{B}}(\lambda)$ such that $\mu = G_\lambda(A) = \{x + JA(x) | x \in \lambda\}$, the graph of the operator A. Note that the boundedness of the operator A is proved by the closed graph theorem and the selfadjointness of A comes from the fact that μ is a Lagrangian subspace. These arguments are same with that of finite dimensional cases. Now we solve the equation

$$L_\mu(u + J(v)) = (\mathcal{P}_\mu + \mathcal{P}_{\lambda^\perp})(u + J(v)) = x + J(y) \tag{2.13}$$

for any given $x, y \in \lambda$ by u and $v, \in \lambda$. Since $\mathcal{P}_\mu(u + J(v)) = x + JA(x)$, $u + J(v) = x + JA(x) + J(a + JA(a)) \in \mu + \mu^\perp$ with an element $a \in \lambda$, and $\mathcal{P}_{\lambda^\perp}(u + J(v)) = J(v) = J(y - A(x))$ we have

$$a = x, \qquad u = x - A(a) = x - A(y) + 2A^2(x), \qquad v = y - A(x).$$

This implies that the operator $L_\mu = \mathcal{P}_\mu + \mathcal{P}_{\lambda^\perp}$ is an isomorphism of H. Note that we have in general $\mathrm{Ker}(\mathcal{P}_\mu + \mathcal{P}_\nu) = \mathrm{Ker}(\mathcal{P}_\mu) \bigcap \mathrm{Ker}(\mathcal{P}_\nu) = \mu^\perp \bigcap \nu^\perp = J(\mu \bigcap \nu)$ (see the proof of Proposition 2.29). □

Remark 2.23. In Proposition 2.29 we will give a generalization of this property after introducing the notion of "*Fredholm pair*".

Proof of Proposition 2.21. Let μ and ν be transversal with $\lambda^\perp$, then

$$\|L_\mu - L_\nu\| = \|\mathcal{P}_\mu - \mathcal{P}_\nu\|. \tag{2.14}$$

So we have

$$L_\nu^{-1} = \sum_{k=0}^{\infty} (L_\mu^{-1}(L_\mu - L_\nu))^k \cdot (L_\mu^{-1})$$

for such ν that $\|L_\mu^{-1}\|\|\mathcal{P}_\mu - \mathcal{P}_\nu\| < 1$, and we have

$$\|L_\nu^{-1}\| \le \sum_{k=0}^{\infty} \|L_\mu^{-1}\|^{k+1}(\|\mathcal{P}_\mu - \mathcal{P}_\nu\|)^k = \|L_\mu^{-1}\| \frac{1}{1 - \|L_\mu^{-1}\|\|\mathcal{P}_\mu - \mathcal{P}_\nu\|}.$$

Hence we have

$$\|L_\mu^{-1} - L_\nu^{-1}\| \le \|L_\mu^{-2}\| \frac{1}{1 - \|L_\mu^{-1}\|\|\mathcal{P}_\mu - \mathcal{P}_\nu\|} \|\mathcal{P}_\mu - \mathcal{P}_\nu\|.$$

Now by putting $x = 0$ in the Eq. (2.13) we have

$$L_\mu^{-1}(J(y)) = -A(y) + J(y) \tag{2.15}$$

and we have the inequality

$$\begin{aligned} \|A_\mu(y) - A_\nu(y)\| &\le \|L_\mu^{-1}(J(y)) - L_\nu^{-1}(J(y))\| \\ &\le \frac{\|L_\mu^{-2}\|}{1 - \|L_\mu^{-1}\|\|\mathcal{P}_\mu - \mathcal{P}_\nu\|} \|\mathcal{P}_\mu - \mathcal{P}_\nu\|\|y\|. \end{aligned}$$

The last inequality shows that the map $G_\lambda^{-1} : \mathbf{O}_{\lambda^\perp} \to \hat{\mathcal{B}}(\lambda)$ is continuous.

The continuity of the map $G_\lambda : \hat{\mathcal{B}}(\lambda) \to \mathbf{O}_{\lambda^\perp}$ is proved more easily: let $\mu \in \mathbf{O}_{\lambda^\perp}$, that is, μ and $\lambda^\perp$ are transversal. Then we can express the element $x + J(y) \in H$ in two ways:

$$\lambda + \lambda^\perp \ni x + J(y) = a + JA(a) + J(b + JA(b)) \in \mu + \mu^\perp.$$

By solving this equation we have

$$a = (\mathrm{Id} + A^2)^{-1}(x + A(y)), \qquad b = (\mathrm{Id} + A^2)^{-1}(y - A(x)),$$

so

$$\mathcal{P}_{G_\lambda(A)}(x + J(y)) = (\mathrm{Id} + A^2)^{-1}(x + A(y)) + JA((\mathrm{Id} + A^2)^{-1}(x + A(y))). \quad (2.16)$$

From this expression of the projection $\mathcal{P}_{G_\lambda(A)}$ and by a standard argument we have

$$\|\mathcal{P}_{G_\lambda(A)} - \mathcal{P}_{G_\lambda(B)}\| \le N(\|A\|, \|B\|)\|A - B\|.$$

Here we denote by $N(s, t)$ a polynomial of degree three of two variables and note that for any $A \in \hat{\mathcal{B}}(\lambda)\|(\mathrm{Id} + A^2)^{-1}\| \le 1$.

Consequently we have proved both of the continuities of the map G_λ and its inverse G_λ^{-1}, in other words, we have proved that the map G_λ gives a local chart of the space $\Lambda(H)$. □

Remark 2.24. From the proof above we see easily that the map G_λ is not isometric.

Proof of Theorem 2.14. It will be clear that a unitary operator which preserves the subspace $\ell^\perp$ comes from an orthogonal transformation on ℓ as the complexification of it. So we have proved Theorem 2.14 together with the help of local sections (2.9) and (2.12). □

Corollary 2.25. *The Lagrangian–Grassmannian $\Lambda(H)$ is an infinite dimensional differentiable manifold modeled on the Banach space of bounded selfadjoint operators.*

Proof. Since we have an open covering $\{\mathbf{O}_{\lambda^\perp}\}_{\lambda\in\Lambda(H)}$ of the Lagrangian–Grassmannian $\Lambda(H)$, each of which is isomorphic to a Banach space $\hat{\mathcal{B}}(\lambda)$, it will be enough to show the coordinate transformations of this covering are "*differentiable*" in a suitable sense. Of course the Banach spaces $\hat{\mathcal{B}}(\lambda)$ are all isomorphic to a typical one.

Let $G_\lambda : \hat{\mathcal{B}}(\lambda) \to \mathbf{O}_{\lambda^\perp}$ be the map in Proposition 2.21, then from the expression of $\mathcal{P}_{G_\lambda(A)}$ (see (2.16)) we know the compositions of maps from $\hat{\mathcal{B}}(\lambda)$ to $\mathcal{B}(H)$

$$\hat{\mathcal{B}}(\lambda) \ni A \mapsto G_\lambda(A) \mapsto \mathcal{P}_{G_\lambda(A)} \mapsto \mathcal{P}_{G_\lambda(A)} + \mathcal{P}_{\mu^\perp} \in \mathcal{B}(H)$$

is a "differentiable" map.

If $G_\lambda(A) \in \mathbf{O}_{\mu^\perp}$, that is, $G_\lambda(A) = G_\mu(B)$ with an operator $B \in \hat{\mathcal{B}}(\mu)$, then from the relation (see (2.15))

$$(\mathcal{P}_{G_\mu(B)} + \mathcal{P}_{\mu^\perp})^{-1}(J(y)) = (\mathcal{P}_{G_\lambda(A)} + \mathcal{P}_{\mu^\perp})^{-1}(J(y)) = -B(y) + J(y)(y \in \mu),$$

it will be apparent that the coordinate transformation $G_\mu^{-1} \circ G_\lambda$: $A \mapsto B = J - (\mathcal{P}_{G_\lambda(A)} + \mathcal{P}_{\mu^\perp})^{-1} \circ J$ is a differentiable map between open sets in Banach space $\hat{\mathcal{B}}(\lambda)$ and $\hat{\mathcal{B}}(\mu)$. □

2.3. Fredholm pairs and Fredholm operators

Theorem 2.14 says that in the infinite dimension we must work in a smaller space than the whole space of Lagrangian subspaces $\Lambda(H)$ to obtain a similar quantity to the Maslov index in the finite dimensional case. In this section we introduce a notion, so called, Fredholm pairs and discuss relations of Fredholm operators and Fredholm pairs [22].

Let ℓ_1 and ℓ_2 be two closed subspaces in H, then first of all, we recall the definition of ℓ_1 and ℓ_2 being a Fredholm pair.

Definition 2.26. We call that two closed subspaces ℓ_1 and ℓ_2 is a Fredholm pair, if

(a) $\dim(\ell_1 \bigcap \ell_2)$ is finite,
(b) $\ell_1 + \ell_2$ is closed and of finite codimensional in H.

We give a relation of two notions "Fredholm pair" and "Fredholm operator".

Proposition 2.27. *Let $\mathcal{P}_1 : H \to H$ be the orthogonal projection operator with the image $\mathcal{P}_1(H) = \ell_1^{\perp}$. Then (ℓ_1, ℓ_2) is a Fredholm pair, if and only if, the restriction $\mathcal{P}_1|_{\ell_2}$ of $\mathcal{P}_1$ to the space ℓ_2 is a Fredholm operator, and*

$$\operatorname{ind} \mathcal{P}_1|_{\ell_2} = \dim \operatorname{Ker} \mathcal{P}_1|_{\ell_2} - \dim \ell_1^{\perp}/\mathcal{P}_1(\ell_2) = \dim \left(\ell_1 \bigcap \ell_2\right) - \dim(H/(\ell_1 + \ell_2)). \tag{2.17}$$

Proof. In the algebraic sense we have $\operatorname{Ker}(\mathcal{P}_1|_{\ell_2}) = \ell_1 \bigcap \ell_2$ and $H/(\ell_1+\ell_2) = \ell_1^{\perp}/\mathcal{P}_1(\ell_2)$ by the definition of the operator $\mathcal{P}_1|_{\ell_2}$. Also we have the closedness of $\ell_1 + \ell_2$ and that $\mathcal{P}_1|_{\ell_2}(\ell_2)$ is equivalent (a little bit general fact is proved in the next Lemma 2.28). These prove the equivalence and we have (2.17). □

Lemma 2.28. *Let $T : H \to H'$ be a bounded surjective operator from a Hilbert space H to a Hilbert space H' and let L be a closed subspace containing* $\operatorname{Ker}(T)$. *Then $T(L)$ is closed.*

Proof. Let π be the orthogonal projection operator in H with the image $= \pi(H) = \operatorname{Ker}(T)$, and let $\tilde{T}$ be an isomorphism between H and $H' \oplus \operatorname{Ker}(T)$ defined by $\tilde{T}(x) = T(x) \oplus \pi(x)$. Then we have $\widetilde{(T)}(L)$ is a closed subspace and we know that $\widetilde{(T)}(L) = T(L) \oplus \operatorname{Ker}(T)$. This implies the closeness of $T(L)$ in H'. □

Next we generalize Lemma 2.22, by which we give a characterization of two Lagrangian subspaces being a Fredholm pair.

Proposition 2.29. *Let $\mu, \nu \in \Lambda(H)$ and let $\mathcal{P}_\mu$ (resp. $\mathcal{P}_\nu$) denote the orthogonal projection operator of H onto μ (resp. ν). Then $\mathcal{P}_\mu + \mathcal{P}_\nu$ is a Fredholm operator, if and only if (μ, ν) is a Fredholm pair.*

Proof. First we show

$$\operatorname{Ker}(\mathcal{P}_\mu + \mathcal{P}_\nu) = \mu^{\perp} \bigcap \nu^{\perp}$$

(see the end of the proof of Lemma 2.22). Since $\mathcal{P}_\mu(x) + \mathcal{P}_\nu(x) = 0$ implies that

$$\langle x, \mathcal{P}_\mu(x)\rangle = \langle x, -\mathcal{P}_\nu(x)\rangle = -\|\mathcal{P}_\nu(x)\|^2 = \|\mathcal{P}_\mu(x)\|^2.$$

Hence $\mathcal{P}_\mu(x) = \mathcal{P}_\nu(x) = 0$, which shows that

$$\mathrm{Ker}(\mathcal{P}_\mu + \mathcal{P}_\nu) = \mu^\perp \bigcap \nu^\perp = J(\mu) \bigcap J(\nu) = J(\mu \bigcap \nu) = (\mu + \nu)^\perp. \tag{2.18}$$

Now let $\mathcal{P}_\mu + \mathcal{P}_\nu$ be a Fredholm operator. Then, since $(\mathcal{P}_\mu + \mathcal{P}_\nu)(H) \subset \mu + \nu$, and $\mathrm{Im}(\mathcal{P}_\mu + \mathcal{P}_\nu)$ is closed and of finite codimensional, so $\mu + \nu$ must be also closed and of finite codimensional. Hence together with the isomorphism (2.18) we have proved that (μ, ν) is a Fredholm pair.

Next assume that (μ, ν) is a Fredholm pair, and we prove $\mathcal{P}_\mu + \mathcal{P}_\nu$ is a Fredholm operator.

From Proposition 2.27 we have $\mathcal{P}_\mu(\nu^\perp)$ (resp. $\mathcal{P}_\nu(\mu^\perp)$) is a finite codimensional closed subspace in μ (resp. ν). Since $\dim(\mu \bigcap \nu) < \infty$, in the direct sum $\mu \oplus \nu$ the subspace $\mathcal{P}_\mu(\nu^\perp) \oplus \mathcal{P}_\nu(\mu^\perp) + \{x \oplus -x | x \in \mu \bigcap \nu\}$ is still closed. Consequently the subspace $\mathcal{P}_\mu(\nu^\perp) + \mathcal{P}_\nu(\mu^\perp)$ is closed and finite codimensional in $\mu + \nu$. Hence the image $(\mathcal{P}_\mu + \mathcal{P}_\nu)(H)$ is a finite codimensional closed subspace in $\mu + \nu$, because it includes the finite codimensional closed subspace $\mathcal{P}_\mu(\nu^\perp) + \mathcal{P}_\nu(\mu^\perp)$. In fact it coincides with $\mu + \nu$, since it is closed and $(\mathcal{P}_\mu + \mathcal{P}_\nu)(H)^\circ = \mu \bigcap \nu$. Now we have proved that $\mathrm{Ker}(\mathcal{P}_\mu + \mathcal{P}_\nu) = J(\mu \bigcap \nu)$ and $\mathrm{Im}(\mathcal{P}_\mu + \mathcal{P}_\nu) = \mu + \nu$, which shows the operator $\mathcal{P}_\mu + \mathcal{P}_\nu$ is a Fredholm operator. □

2.4. *Fredholm–Lagrangian–Grassmannian*

We fix a Lagrangian subspace λ and introduce a subspace of $\Lambda(H)$, we call, *Fredholm–Lagrangian–Grassmannian* with respect to λ.

Definition 2.30. The Fredholm–Lagrangian–Grassmannian of H with respect to a Lagrangian subspace λ is defined as

$$\mathcal{F}\Lambda_\lambda(H) = \{\mu \in \Lambda(H) | (\mu, \lambda) \ \text{ is a Fredholm pair}\}. \tag{2.19}$$

Definition 2.31. We call the subset

$$\mathfrak{M}_\lambda(H) = \left\{\mu \in \mathcal{F}\Lambda_\lambda(H) | \mu \bigcap \lambda \neq \{0\}\right\}, \tag{2.20}$$

the Maslov cycle with respect to λ.

Notation 2.32. $\mathcal{F}\Lambda^{(0)}_\lambda(H) = \{\theta \in \mathcal{F}\Lambda_\lambda(H) | \theta \text{ is transversal to } \lambda\} = \mathcal{F}\Lambda_\lambda(H) \setminus \mathfrak{M}_\lambda(H)$ ($= \mathbf{O}_\lambda$, see Notation 2.19).

Remark 2.33.

(a) In the finite dimensional case, the subset $\mathfrak{M}_\lambda(H)$ is a singular cycle whose homology class is a generator of the codimension one homology group $H_{(n(n+1)/2)-1}(\Lambda(H), \mathbb{Z})$, where we put $\dim H = 2n$.
(b) As we proved in Proposition 2.21 the subset $\mathcal{F}\Lambda_\lambda(H) \setminus \mathfrak{M}_\lambda(H) = \mathcal{F}\Lambda^{(0)}_\lambda(H)$ is isomorphic to the space of bounded selfadjoint operators on $\lambda^\perp$.

First we study how the Fredholm–Lagrangian–Grassmannian $\mathcal{F}\Lambda_\lambda(H)$ depends on the space λ. In the finite dimensional case, it is clear that $\mathcal{F}\Lambda_\lambda(H) = \Lambda(H)$. In the infinite dimension, $\mathcal{F}\Lambda_\lambda(H)$ is an open subset of $\Lambda(H)$. Openness follows from Propositions C.2 and 2.29, and it cannot include λ itself. However we can prove the following proposition.

Proposition 2.34. *λ can be approximated by a sequence in $\mathcal{F}\Lambda_\lambda(H)$, i.e., $\lambda \in \partial\mathcal{F}\Lambda_\lambda(H)$ (= the boundary).*

Proof. Let $A : \lambda \to \lambda$ be a bounded selfadjoint operator and assume that A is an isomorphism. Then for all $\epsilon > 0$, the Lagrangian subspace

$$G_{\epsilon\cdot A} = \{x + \epsilon JA(x) | x \in \lambda\}$$

is transversal with both of λ and $\lambda^\perp$. Since ϵA converges to 0 in $\hat{\mathcal{B}}(\lambda)$ when $\epsilon \to 0$, we know that the orthogonal projection operator $\mathcal{P}_{G_{\epsilon\cdot A}}$ onto the graph of $\epsilon \cdot J \circ A$ converges to $\mathcal{P}_\lambda$. Hence we have

$$\lambda \in \overline{\mathcal{F}\Lambda_\lambda(H)} \setminus \mathcal{F}\Lambda_\lambda(H). \qquad \square \tag{2.21}$$

Let λ and μ in $\Lambda(H)$ and assume that

$$\mu = U(\lambda) \text{ with } U = \mathrm{Id} + K \in \mathcal{U}(H_J) \text{ is of the form Id} + \textit{compact operator}, \tag{2.22}$$

then the following proposition holds.

Proposition 2.35.

$$\mathcal{F}\Lambda_\lambda(H) = \mathcal{F}\Lambda_\mu(H).$$

Proof. By Proposition 2.29, $\nu \in \mathcal{F}\Lambda_\lambda(H)$ if and only if $\mathcal{P}_\lambda + \mathcal{P}_\nu$ is a Fredholm operator. From the assumption, $\mathcal{P}_\mu + \mathcal{P}_\nu = \mathcal{P}_{U(\lambda)} + \mathcal{P}_\nu = U \circ \mathcal{P}_\lambda \circ U^{-1} + \mathcal{P}_\nu = (\mathrm{Id} + K) \circ \mathcal{P}_\lambda \circ (\mathrm{Id} + K^*) + \mathcal{P}_\nu = \mathcal{P}_\lambda + \mathcal{P}_\nu +$ compact operator. Hence if $\nu \in \mathcal{F}\Lambda_\lambda(H)$, then $\nu \in \mathcal{F}\Lambda_\mu(H)$. Since $(\mathrm{Id} + K)^{-1} = \mathrm{Id} + K^*$, by the same way we have $\mathcal{F}\Lambda_\mu(H) \subset \mathcal{F}\Lambda_\lambda(H)$. $\square$

Definition 2.36. We denote by $\mathcal{U}_{\mathrm{res}}(H_J)$ the subgroup of $\mathcal{U}(H_J)$ consisting of such operators that

$$\mathcal{U}_{\mathrm{res}}(H_J) = \{\mathrm{Id} + \text{compact operator}\}.$$

Corollary 2.37. *The group $\mathcal{U}_{\mathrm{res}}(H_J)$ acts on $\mathcal{F}\Lambda_\lambda(H)$ and $\mathcal{U}_{\mathrm{res}}(H_J)(\lambda) \subset \partial(\mathcal{F}\Lambda_\lambda(H))$, that is, the orbit of the element λ is also included in the boundary of $\mathcal{F}\Lambda_\lambda(H)$.*

As a special case of the relation (2.22) we introduce an equivalence relation on the space $\Lambda(H)$.

Definition 2.38. We call λ and $\mu \in \Lambda(H)$ almost coincide, if

$$\dim(\lambda/(\lambda + \mu)) < +\infty$$

and denote

$$\lambda \sim \mu,$$

when two Lagrangian subspaces λ and μ almost coincide.

It will be easy to prove that this is in fact an equivalence relation. Note that in this case

$$\dim(\lambda/(\lambda+\mu)) = \dim(\mu/(\lambda+\mu)),$$

and in fact we have the following proposition.

Proposition 2.39. *Let $\lambda \sim \mu$, then there exists a unitary operator U of the form* $U = \mathrm{Id} + \text{compact operator}$ *such that* $\mu = U(\lambda)$.

Proof. Since λ and μ are Lagrangian subspaces, the sum of the complex subspaces spanned by $\lambda \bigcap \mu$ and $\lambda \bigcap (\lambda \bigcap \mu)^\perp$ is an orthogonal sum of H_J, and so $(\lambda \bigcap (\lambda \bigcap \mu)^\perp) \otimes \mathbb{C} = (\mu \bigcap (\lambda \bigcap \mu)^\perp) \otimes \mathbb{C}$ in H_J. Hence we can find such an unitary operator U that is identity on the subspace $(\lambda \bigcap \mu) \otimes \mathbb{C}$. Hence we can take $U = \mathrm{Id} + K$ with K being a finite rank operator. □

Proposition 2.40. *Let $\lambda \in \Lambda(H)$ and let $W \subset \lambda$ be a finite codimensional closed subspace in λ. Then for $\mu \in \Lambda(H)$, the pair (λ, μ)* is a Fredholm pair, if and only if, (W, μ) is a Fredholm pair.

We denote by $\mathcal{F}\Lambda_W(H)$

$$\mathcal{F}\Lambda_W(H) = \{\mu \in \Lambda(H) | (W, \mu) \text{ is a Fredholm pair}\}. \tag{2.23}$$

Proof of Proposition 2.40. We prove $\mathcal{F}\Lambda_W(H) = \mathcal{F}\Lambda_\lambda(H)$.

Let $\mu \in \mathcal{F}\Lambda_W(H)$. Then, since the map

$$\left(\lambda \bigcap \mu\right) / \left(W \bigcap \mu\right) \to \lambda / W$$

is injective, we have

$$\dim\left(\lambda \bigcap \mu\right) \le \dim(\lambda/W) + \dim\left(W \bigcap \mu\right),$$

and the space $\lambda + \mu$ is a finite dimensional extension of the closed subspace $W + \mu$. Hence λ and μ is a Fredholm pair.

Now let $\mu \in \mathcal{F}\Lambda_\lambda(H)$. In the short exact sequence

$$0 \to \lambda \bigcap \mu \xrightarrow{j} \lambda \oplus \mu \xrightarrow{\tau} \lambda + \mu \to 0,$$

where $j(a) = a \oplus -a \in H \oplus H$ and $\tau(a \oplus b) = a + b$, we have

$$\tau^{-1}(W + \mu) = W \oplus \mu + j\left(\lambda \bigcap \mu\right).$$

Hence $W + \mu$ must be closed in $\lambda + \mu$, so is in H. Also we have $\dim W \bigcap \mu < \infty$. These proves the coincidence $\mathcal{F}\Lambda_W(H) = \mathcal{F}\Lambda_\lambda(H)$. □

Corollary 2.41. *If $\lambda \sim \mu$, then $\mathcal{F}\Lambda_\lambda(H) = \mathcal{F}\Lambda_\mu(H)$.*

Proof. By Proposition 2.39 we already know this, but also by putting $W = \lambda \bigcap \mu$ in the proof of Proposition 2.40 we can prove the coincidence. □

Remark 2.42. Since in the proof of the above proposition we did not use any particular properties of Lagrangian subspaces, the above coincidence holds for any Fredholm pair (L_1, L) and (L_2, L), where L_2 is a finite codimensional closed subspace in L_1.

Finally we note the following proposition.

Proposition 2.43. *We have an open covering $\mathcal{F}\Lambda_\lambda(H) = \bigcup_{\mu\sim\lambda} \mathbf{O}_\mu$, and each $\mathbf{O}_\mu$ is open dense in $\mathcal{F}\Lambda_\lambda(H)$. Hence $\bigcap_{i=1}^{\infty} \mathbf{O}_{\mu_i}$ (each $\mu_i \sim \lambda$) is not empty. In other words, for any given countable number of Lagrangian subspaces $\{\mu_i\}_{i=1}^{\infty}$ such that each of which is equivalent to a fixed Lagrangian subspace λ, there exists a Lagrangian subspace which is transversal to each μ_i.*

2.5. *Souriau map and the universal Maslov cycle*

When we fix a $\lambda \in \Lambda(H)$ then we have an identification

$$H_J = \lambda \oplus \lambda^\perp = \lambda \oplus J\lambda \cong \lambda \otimes \mathbb{C}, \qquad x + Jy \mapsto x \otimes 1 + y \otimes \sqrt{-1}. \tag{2.24}$$

We denote by τ_λ the complex conjugation in H_J under this identification:

$$\tau_\lambda(x + J(y)) = x - J(y), \quad x, y \in \lambda.$$

It will be easy to show the following relation:

$$\tau_\lambda = 2\mathcal{P}_\lambda - \mathrm{Id}. \tag{2.25}$$

Any $U \in \mathcal{U}(H_J)$ can be expressed as

$$U = X + \sqrt{-1}Y$$

with $X, Y \in \mathcal{B}(\lambda)$ in such a way that

$$\begin{aligned} U(x \otimes 1 + y \otimes \sqrt{-1}) &= (X(x) - Y(y)) \otimes 1 + (X(y) + Y(x)) \otimes \sqrt{-1} \\ &= X(x) - Y(y) + J(X(y) + Y(x)), \end{aligned}$$

and X, Y satisfy the relations:

$$X^tY = Y^tX, \qquad {}^tYX = {}^tXY,$$

$$X^tX + Y^tY = \mathrm{Id}, \qquad {}^tXX + {}^tYY = \mathrm{Id}.$$

For $\lambda \in \Lambda(H)$ we denote by θ_λ an anti-group isomorphism $\mathcal{U}(H_J) \to \mathcal{U}(H_J)$ defined by

$$\theta_\lambda(U) = \tau_\lambda \circ U^* \circ \tau_\lambda, \tag{2.26}$$

then $\theta_\lambda(U) = {}^tX + \sqrt{-1}\,{}^tY$.

Note that if $Y \neq 0$, $\theta_\lambda(U) \neq {}^tU$, where we mean ${}^tU$ is a transposed operator when we regard U as a real linear operator.

Then we have

$$\{U \in \mathcal{U}(H_J) | \theta_\lambda(U) = U^{-1}\} = \mathcal{O}(\lambda) (= \text{orthogonal group on } \lambda). \tag{2.27}$$

Hence the map $\mathcal{U}(H_J) \to \mathcal{U}(H_J)$, $U \mapsto U \circ \theta_\ell(U)$ induces a continuous map (see Corollary 2.16)

$$\mathcal{S}_\ell : \Lambda(H) \to \mathcal{U}(H_J), \qquad \mu = U(\ell^\perp) \mapsto \mathcal{S}_\ell(\mu) = U \circ \theta_\ell(U). \tag{2.28}$$

We call this map as “*Souriau map*” henceforth.

From the relation (2.25) we have an expression of the Souriau map in terms of projection operators corresponding to Lagrangian subspaces.

Proposition 2.44. $\mathcal{S}_\ell(\mu) = (\mathrm{Id} - 2\mathcal{P}_\mu)(2\mathcal{P}_\ell - \mathrm{Id}) = -\tau_\mu \circ \tau_\ell$.

Corollary 2.45. *Let* λ, μ, ν *be three Lagrangian subspaces, then*

$$\mathcal{S}_\mu(\nu) \circ \mathcal{S}_\lambda(\mu) = -\mathcal{S}_\lambda(\nu). \tag{2.29}$$

From the relations (2.25), (2.26) and Proposition 2.44 we have the following proposition.

Proposition 2.46. *The maps*

$$\begin{aligned} &\mathcal{U}(H_J) \times \Lambda(H) \to \mathcal{U}(H_J), \\ &(\ell, U) \mapsto U \circ \theta_\ell(U) = U \circ (2\mathcal{P}_\lambda - \mathrm{Id}) \circ U^* \circ (2\mathcal{P}_\lambda - \mathrm{Id}), \end{aligned} \tag{2.30}$$

and

$$\Lambda(H) \times \Lambda(H) \to \mathcal{U}(H_J), \qquad (\ell, \mu) \mapsto \mathcal{S}_\ell(\mu) = (\mathrm{Id} - 2\mathcal{P}_\mu)(2\mathcal{P}_\ell - \mathrm{Id}) \tag{2.31}$$

are continuous.

By Proposition 2.44

$$U \circ \mathcal{S}_\ell(\mu) \circ U^* = \mathcal{S}_{U(\ell)}(U(\mu)), \tag{2.32}$$

that is, the following diagram is commutative.

Proposition 2.47.

$$\begin{array}{ccc} \Lambda(H) & \xrightarrow{\mathcal{S}_\ell} & \mathcal{U}(H_J) \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle Ad_U} \\ \Lambda(H) & \xrightarrow[\mathcal{S}_{U(\ell)}]{} & \mathcal{U}(H_J). \end{array} \tag{2.33}$$

In particular, when $U \in \mathcal{U}_{\mathrm{res}}(H_J)$ we have a commutative diagram.

Proposition 2.48.

$$\begin{array}{ccc} \mathcal{F}\Lambda_\ell(H) = \mathcal{F}\Lambda_{U^{-1}(\ell)}(H) & \xrightarrow{\mathcal{S}_{U^{-1}(\ell)}} & \mathcal{U}(H_J) \\ {\scriptstyle U}\big\downarrow & & \big\downarrow{\scriptstyle Ad_U} \\ \mathcal{F}\Lambda_\ell(H) & \xrightarrow[\mathcal{S}_\ell]{} & \mathcal{U}(H_J). \end{array} \tag{2.34}$$

Here we remark the adjoint operator of an anti-linear operator: let T be an anti-linear operator on a complex Hilbert space H with a Hermitian inner product $(\bullet, \bullet)$, then the adjoint operator T^* is defined by the relation $(T(z), w) = (T^*(w), z)$ $(z, w \in H)$. Then T^* is again an anti-linear operator and we have a composition formula with a linear or anti-linear operator L: $(T \circ L)^* = L^* \circ T^*$.

Now τ_λ is anti-linear and we have by a direct calculation

$$\tau_\lambda^* = \tau_\lambda, \quad \text{that is } (\tau_\lambda(z), w)_J = (\tau_\lambda(w), z)_J.$$

From this fact $\theta_\lambda^2 = \mathrm{Id}$, in other words, θ_λ is an anti-linear involution on $\mathcal{B}(H_J)$.

By the above remark and the expression of the Souriau map (Proposition 2.44) we have the following proposition.

Proposition 2.49.

$$\mathcal{S}_\lambda(\mu)^* = \mathcal{S}_\mu(\lambda). \tag{2.35}$$

We call the restriction of the Souriau map to $\mathcal{F}\Lambda_\lambda(H)$ also Souriau map always.

Now for a fixed λ, we put $\mathcal{U}_\lambda(H_J) = \rho_\lambda^{-1}(\mathcal{F}\Lambda_\lambda(H))$, where $\rho_\lambda : \mathcal{U}(H_J) \to \mathcal{F}\Lambda_\lambda(H)$, $\rho_\lambda(U) = U(\lambda^\perp)$.

Proposition 2.50. *Let $U \in \mathcal{U}(H_J)$, then $U = X + \sqrt{-1}Y \in \mathcal{U}_\lambda(H_J)$, if and only if, $X \in \mathcal{B}(\lambda)$ is a Fredholm operator.*

Proof. Let $U \in \mathcal{U}(H_J)$, and put $\mu = U(\lambda^\perp)$. Then the inclusion map $\lambda^\perp \to H = \lambda + \lambda^\perp$ induces the isomorphism

$$(\lambda + \lambda^\perp)/(\lambda + \mu) \cong \lambda^\perp / J(X(Y^{-1}(\lambda))) = \lambda^\perp / J(X(\lambda)) \cong \lambda / X(\lambda).$$

Also

$$\lambda \cap U(\lambda^\perp) \cong \mathrm{Ker}\, X.$$

These shows the assertions. □

Let $\mu \in \mathcal{F}\Lambda_\lambda(H)$ and $U(\lambda^\perp) = \mu$, then by the definition of the Souriau map $\mathcal{S}_\lambda(\mu) = U \circ \theta_\lambda(U)$ and from above Proposition 2.50, we have the following proposition.

Proposition 2.51.

$$U \circ \theta_\lambda(U) + \mathrm{Id} \tag{2.36}$$

is a Fredholm operator.

Proof. Let $U = X + \sqrt{-1}Y$, with $X, Y \in \mathcal{O}(\lambda)$, then

$$U \circ \theta_\lambda(U) + \mathrm{Id} = 2X \circ \theta_\lambda(U),$$

and this shows the Fredholmness of the operator $U \circ \theta_\lambda(U) + \mathrm{Id}$. □

Let $\mu \in \Lambda(H)$, then from the relation that an element $z = x + J(y)$ $(x, y \in \lambda)$ is in μ if and only if $-z = W_\mu(\tau_\lambda(z))$, we have

$$\mathrm{Ker}(W_\mu + \mathrm{Id}) = (\mu \cap \lambda) \otimes \mathbb{C} \cong (\mu \cap \lambda) \oplus J(\mu \cap \lambda). \tag{2.37}$$

Hence

Proposition 2.52. *For any $\mu \in \mathcal{F}\Lambda_\lambda(H)$ and any $U \in \mathcal{U}_\lambda(H_J)$ with $\mu = U(\lambda^\perp)$*

$$\dim_{\mathbb{R}}(\mu \cap \lambda) = \dim_{\mathbb{C}} \mathrm{Ker}(W_\mu + \mathrm{Id}).$$

Let us now consider the space

$$\mathcal{U}_{\mathcal{F}}(H_J) = \{U \in \mathcal{U}(H_J) | U + \mathrm{Id} \text{ is a Fredholm operator}\} \tag{2.38}$$

and a subset

$$\mathcal{U}_{\mathfrak{M}}(H_J) = \{U \in \mathcal{U}_{\mathcal{F}}(H_J) | \mathrm{Ker}(U + \mathrm{Id}) \neq \{0\}\}, \tag{2.39}$$

which by the preceding Proposition 2.52 we can regard as a kind of the universal Maslov cycle.

Proposition 2.53. *For any λ, $\mathcal{S}_\lambda^{-1}(\mathcal{U}_{\mathfrak{M}}(H_J)) = \mathfrak{M}_\lambda(H)$.*

Now we state the fundamental property for discussing the Maslov index in the infinite dimension.

Theorem 2.54.

(a) $\pi_1(\mathcal{F}\Lambda_\lambda(H)) \cong \mathbb{Z}$,
(b) $\pi_1(\mathcal{U}_{\mathcal{F}}(H_J)) \cong \mathbb{Z}$,
(c) *The induced map*

$$(\mathcal{S}_\lambda)_* : \pi_1(\mathcal{F}\Lambda_\lambda(H)) \to \pi_1(\mathcal{U}_{\mathcal{F}}(H_J))$$

is an isomorphism.

We give the proof of this theorem in the next subsection by the method of the finite dimensional reduction.

2.6. Proof of Theorem 1.54(a)

Notation 2.55.

(a) Let $\lambda \in \Lambda(H)$. We denote by $\mathrm{Sub}_{\mathrm{fin}}(\lambda)$ the set of all closed subspaces $W \subset \lambda$ $(W \neq \lambda)$ of finite codimensions.

(b) Let W be a closed isotropic subspace such that $\dim W^\circ/W < \infty$. We denote by $\Lambda(W, H)$ the set of Lagrangian subspaces of H which contains W.

Theorem 2.56. *Let* $\lambda \in \Lambda(H)$ *and* $W \in \mathrm{Sub}_{\mathrm{fin}}(\lambda)$:

(a) *The inclusions*

$$\mathcal{F}\Lambda_W^{(0)} = \{\theta \in \mathcal{F}\Lambda_\lambda(H) | \theta \cap W = \{0\}\} \hookrightarrow \mathcal{F}\Lambda_\lambda(H),$$

define an isomorphism

$$\mathrm{ind} - \lim_{W \to \{0\}} \pi_1(\mathcal{F}\Lambda_W^{(0)}(H)) \xrightarrow{\sim} \pi_1(\mathcal{F}\Lambda_\lambda(H)).$$

(b) *There is a natural isomorphism*

$$\pi_1(\mathcal{F}\Lambda_W^{(0)}(H)) \xrightarrow{\sim} \pi_1(\Lambda(W^\circ/W)) \cong \mathbb{Z}$$

for each $W \in \mathrm{Sub}_{\mathrm{fin}}(\lambda)$.

By combining (a) and (b) we obtain Theorem 2.54(a).

The proof of Theorem 2.56 will follow from two Propositions below which will be of independent interest. First we shall prove the following proposition.

Proposition 2.57. *Let* $K \subset \mathcal{F}\Lambda_\lambda(H)$ *be a compact subset. Then there exists a* $W \in \mathrm{Sub}_{\mathrm{fin}}(\lambda)$ *such that* $\mu \cap W = \{0\}$ *for all* $\mu \in K$.

Proof. Let $\mu_0 \in K$. The sum of the orthogonal projections $\mathcal{P}_\lambda + \mathcal{P}_{\mu_0}$ is a Fredholm operator by Proposition 2.29 and we have

$$\mathrm{Ker}(\mathcal{P}_\lambda + \mathcal{P}_{\mu_0}) = J(\lambda \cap \mu_0).$$

Let

$$h = (J(\lambda \cap \mu_0))^\perp = \lambda + (\lambda^\perp \cap (J(\lambda \cap \mu_0))^\perp).$$

Then the operator $\mathcal{P}_\lambda + \mathcal{P}_{\mu_0}$ is injective on h and its range $\lambda + \mu_0$ is closed. Hence there exists an open neighborhood U of μ_0 in $\mathcal{F}\Lambda_\lambda(H)$ such that $\mathcal{P}_\lambda + \mathcal{P}_\mu$ is injective on h for all $\mu \in K \cap U$. Since K compact, a finite set $U_1, \dots, U_N$ of such neighborhoods covers the whole of K. Then

$$W = \bigcap_{j=1}^{N} ((\lambda \cap \mu_j)^\perp \cap \lambda)$$

satisfies our requirement for suitable choices of $\mu_j \in U_j \cap K$. □

The next proposition gives us a property of $\mathcal{F}\Lambda_\lambda(H)$ relating with the finite dimensional reduction of the Maslov index.

Proposition 2.58. *Let $W \in \mathrm{Sub}_{\mathrm{fin}}(\lambda)$, then the mapping*

$$\rho_W : \mathcal{F}\Lambda_W^{(0)}(H) \to \Lambda(W^\circ/W), \qquad \mu \mapsto ((\mu \cap W^\circ) + W)/W$$

defines a fiber bundle.

The proof of this proposition is given by proving two lemmas below.

Lemma 2.59.

(a) *Let H, λ, W be as above and let $\theta \in \Lambda(W, H)$, i.e., θ is a Lagrangian subspace including W. Then*

$$U_\theta = \{\mu \in \mathcal{F}\Lambda_W^{(0)}(H) | \mu \cap \theta = \{0\}\}$$

is an open subset of the total space $\mathcal{F}\Lambda_W^{(0)}(H)$ and we have

$$\bigcup_{\theta \in \Lambda(W,H)} U_\theta = \mathcal{F}\Lambda_W^{(0)}(H).$$

(b) *Let $\bar{\theta} = \theta/W \in \Lambda(W^\circ/W)$. Then the set*

$$U_{\bar{\theta}} = \{\ell \in \Lambda(W^\circ/W) | \ell \cap \bar{\theta} = \{0\}\}$$

is an open subset of the Lagrangian–Grassmannian manifold $\Lambda(W^\circ/W)$ and the union of all such subsets covers $\Lambda(W^\circ/W)$.

(b) *The mapping*

$$\rho_W : U_\theta \to U_{\bar{\theta}}$$

is surjective.

Proof. Since $\mu \in U_\theta$ is transversal with θ, openness of U_θ follows from Lemma 2.22. For a given $\mu \in \mathcal{F}\Lambda_W^{(0)}(H)$ one finds easily a $\theta = W + L \in \Lambda(H)$ with $\theta \cap \mu = \{0\}$, by taking a suitable Lagrangian subspace L in $(\lambda \cap W^\perp) \oplus J(\lambda \cap W^\perp)$. That gives the claimed open covering and (b) and (c) can be seen easily. □

Again let $W \in \mathrm{Sub}_{\mathrm{fin}}(\lambda)$ and $\theta \in \mathcal{F}\Lambda_\lambda(H)$, $\theta \supset W$ and we decompose H into four mutually orthogonal subspaces:

$$H = \theta + J(\theta) = W^\perp \cap \theta + W + J(W^\perp \cap \theta) + J(W). \tag{2.40}$$

Lemma 2.60. *Let $\mu \in U_\theta$. Then there exist linear mappings*

$$a : J(W^\perp \cap \theta) \to W^\perp \cap \theta$$

and

$$g : J(W^\perp \cap \theta) \to W$$

such that each $z \in \mu \cap W^0$ can be written in the form

$$z = x + a(x) + g(x) \quad \text{with} \;\; x \in J(W^{\perp} \cap \theta).$$

Proof. Since μ intersects θ transversally, there is a map $A : J(\theta) \to \theta$ such that $A \circ J$ selfadjoint on θ and $\mu = \{u + Au | u \in J(\theta)\}$. We decompose $u = x + y$ with $x \in J(W^{\perp} \cap \theta)$ and $y \in J(W)$ according to the decomposition of $J(\theta)$ in (2.40). With regard of that decomposition, the mapping A can be written as a 2×2 matrix

$$\begin{pmatrix} a & b \\ g & d \end{pmatrix}.$$

More explicitly, we have

$$Au = a(x) + b(y) + g(x) + d(y),$$

where

$$\begin{aligned} &a : J(W^{\perp} \cap \theta) \to W^{\perp} \cap \theta, \qquad b : J(W) \to W^{\perp} \cap \theta, \\ &g : J(W^{\perp} \cap \theta) \to W, \qquad d : J(W) \to W. \end{aligned}$$

We notice that

$$a \circ J, \text{ and } d \circ J \text{ are selfadjoint, and } {}^t(b \circ J) = g \circ J. \tag{2.41}$$

Now, let $z \in \mu \cap W^0$. It can be written as

$$z = u + Au = x + y + a(x) + b(y) + g(x) + d(y).$$

From the decomposition (2.40) it follows that the component y in $J(W)$ must vanish. So

$$z = x + a(x) + g(x). \qquad \square$$

Corollary 2.61. *Let λ, W, θ be as above. Let $\mu = \{u + Au | u \in J(\theta)\} \in U_{\theta}$ with*

$$A = \begin{pmatrix} a & b \\ g & d \end{pmatrix}$$

with respect to the decompositions $J(\theta) = J(W^{\perp} \cap \theta) + J(W)$ and $\theta = W^{\perp} \cap \theta + W$. As before, we identify W°/W with $(W^{\perp} \cap \theta) + J(W^{\perp} \cap \theta)$. Then

$$\rho_W(\mu) = \{x + a(x) | x \in J(W^{\perp} \cap \theta)\}. \tag{2.42}$$

In particular, two $\mu, \mu' \in U_{\theta}$ belong to the same fiber, i.e., $\rho_W(\mu) = \rho_W(\mu')$, if and only if, $a = a'$.

Now we prove Proposition 2.58.

Proof. We define a local trivialization on $U_{\bar{\theta}}$:

$$\begin{array}{ccc} U_{\bar{\theta}} \times F & \xrightarrow{\tau} & U_{\theta} \\ & \pi \searrow & \downarrow \rho_W \\ & & U_{\bar{\theta}} \end{array} \tag{2.43}$$

Here, π denotes the projection onto the first component. We take

$$F = \mathcal{B}(J(W),\, W^{\perp} \cap \theta) + \mathcal{B}_{sa}(J(W),\, W),$$

where $\mathcal{B}(J(W),\, W^{\perp} \cap \theta)$ denotes the vector space of bounded operators from $J(W)$ to $W^{\perp} \cap \theta$ and $\mathcal{B}_{sa}(J(W),\, W)$ the vector space of bounded operators from $J(W)$ to W each of which operator becomes a selfadjoint operator on W by combing with J. For a fixed point $L \in U_{\bar{\theta}}$ and a point in the fiber $(b, d) \in F$, we define

$$\tau(L; b, g) = \left\{ u + Au | A = \begin{pmatrix} a_L & b \\ g_b & d \end{pmatrix}, u \in J(\theta) \right\}$$

with the decomposition $J(\theta) = J(W^{\perp} \cap \theta) + J(W)\}$. The operator $a_L : J(W^{\perp} \cap \theta) \to W^{\perp} \cap \theta$ with $a_L \circ J$ selfadjoint is uniquely determined by the condition

$$L = \{x + a_L(x) | x \in J(W^{\perp} \cap \theta)\}.$$

As a consequence, the map τ is surjective and injective. By the definition of a_L from L we have the commutativity of the diagram (2.43). □

Before proving Theorem 2.56 we remark the a commutative diagram (2.44) below.
Let us consider two spaces $W, W' \in \mathrm{Sub}_{\mathrm{fin}}(\lambda)$ with $W' \subset W$. So

$$\mathcal{F}\Lambda_W(H) = \mathcal{F}\Lambda_{W'}(H) = \mathcal{F}\Lambda_{\lambda}(H)$$

and

$$\mathcal{F}\Lambda_W^{(0)} \subset \mathcal{F}\Lambda_{W'}^{(0)}(H) \subset \mathcal{F}\Lambda_{\lambda}(H).$$

Recall that $\Lambda(W, H)$ denotes the set of Lagrangian subspaces of H which contain W, and then this space is isomorphic with the Lagrangian–Grassmannian $\Lambda(W^{\circ}/W)$ in an obvious way:

$$\Lambda(W, H) \xrightarrow{\sim} \Lambda(W^{\circ}/W), \qquad \theta \mapsto \theta/W,$$

and a corresponding isomorphism for W'. Now let $C : I \to \mathcal{F}\Lambda_{\lambda}(H)$ be a curve which is transversal to W. So, it gives us the curve $C : I \to \mathcal{F}\Lambda_W^{(0)}(H)$. Then we have the following commutative diagram:

$$
\begin{array}{ccc}
I & \xrightarrow{C} & \mathcal{F}\Lambda_\lambda(H) \\
{\scriptstyle C}\big\downarrow & & \big\uparrow{\scriptstyle \cup} \\
\mathcal{F}\Lambda_W^{(0)}(H) & \underset{\hookrightarrow}{\longrightarrow} & \mathcal{F}\Lambda_{W'}^{(0)}(H) \\
{\scriptstyle \rho_W}\big\downarrow & & \big\downarrow{\scriptstyle \rho_{W'}} \\
\Lambda(W^\circ/W) & \longrightarrow & \Lambda(W'^\circ/W') \\
{\scriptstyle \cong}\big\uparrow & & \big\uparrow{\scriptstyle \cong} \\
\Lambda(W,H) & \underset{\mathbf{h}_{W,W'}}{\longrightarrow} & \Lambda(W',H)\,.
\end{array}
\tag{2.44}
$$

Proof of Theorem 2.56. By Proposition 2.57, it will not be difficult to see the mapping

$$\underset{W\to\{0\}}{\text{ind-lim}}\ \pi_1(\mathcal{F}\Lambda_W^{(0)}(H)) \to \pi_1(\mathcal{F}\Lambda_\lambda(H))$$

is naturally isomorphic (and also it is isomorphic for all homotopy groups, but we do not treat with higher homotopy groups).

To see (b), we just notice that the maps $\mathbf{h}_{W,W'}$ in the above commutative diagram gives us isomorphisms of their fundamental groups [1] together with the help of the exact sequence

$$\{0\} = \pi_1(F) \to \pi_1(\mathcal{F}\Lambda_W^{(0)}(H)) \underset{\mathbf{p}_{W*}}{\longrightarrow} \pi_1((W^\circ/W)) \to \pi_0(F) = \{0\}. \qquad \square$$

2.7. *Proof of Theorem 1.54(b) and (c)*

In this section first we explain the space $\mathcal{U}_{\mathcal{F}}(H_J)$ in the framework of the complexified symplectic Hilbert space (Proposition 2.64) and give a proof of the isomorphisms.

Proposition 2.62. $\pi_1(\mathcal{F}\Lambda_\lambda(H)) \underset{(\mathcal{S}_\lambda)_*}{\overset{\sim}{\longrightarrow}} \pi_1(\mathcal{U}_{\mathcal{F}}(H_J)) \cong \mathbb{Z}$.

Then these will give a proof of 2.54(b) *and* (c).

Let H be a separable symplectic Hilbert space with compatible symplectic form ω, an inner product $\langle \bullet, \bullet \rangle$ and an almost complex structure J, $\omega(x, y) = \langle J(x), y\rangle$, $J^2 = -\mathrm{Id}$. The complexification $H \otimes \mathbb{C}$ of the real Hilbert space is installed with the Hermitian inner product as usual and we denote by $\Lambda^{\mathbb{C}}(H \otimes \mathbb{C})$ the space of complex Lagrangian subspaces in $H \otimes \mathbb{C}$:

$$\Lambda^{\mathbb{C}}(H \otimes \mathbb{C}) = \{\ell | \ell \text{ is a complex subspace such that } \ell^\perp = J(\ell)\}.$$

Then a subgroup of the unitary operators in $H \otimes \mathbb{C}$, we denote it by $\mathcal{U}_0(H \otimes \mathbb{C})$, consisting of those operators U that $U(\ell)^\perp = J(U(\ell))$ for any $\ell \in \Lambda^{\mathbb{C}}(H \otimes \mathbb{C})$ acts on $\Lambda^{\mathbb{C}}(H \otimes \mathbb{C})$ transitively. This condition for $U \in \mathcal{U}_0(H \otimes \mathbb{C})$ is equivalent to say that it commutes with the complexified almost complex structure J.

Taking the complexification of $\lambda \in \Lambda(H)$ gives us a natural embedding $\Lambda(H) \to \Lambda^{\mathbb{C}}(H \otimes \mathbb{C})$, and its restriction to $\mathcal{F}\Lambda_\lambda(H)$ has the image in $\mathcal{F}\Lambda^{\mathbb{C}}_{\lambda\otimes\mathbb{C}}(H \otimes \mathbb{C})$, a subspace of $\Lambda^{\mathbb{C}}(H \otimes \mathbb{C})$ consisting of those subspaces which are Fredholm pairs with $\lambda \otimes \mathbb{C}$. We denote this map by $\otimes\mathbb{C}$.

When we consider an operator $U \in \mathcal{U}(H_J)$ as a real operator and take its complexification, we denote it by $U^{\mathbb{C}}$, then $U^{\mathbb{C}}$ is in $\mathcal{U}_0(H \otimes \mathbb{C})$ and we have $U(\mu) \otimes \mathbb{C} = U^{\mathbb{C}}(\mu \otimes \mathbb{C})$, $\mu \in \Lambda(H)$, and the following diagram is commutative:

$$\begin{tikzcd} \mathcal{U}(H_J) \arrow[r, "U\mapsto U^{\mathbb{C}}"] \arrow[d, "\rho_\ell"'] & \mathcal{U}(H \otimes \mathbb{C}) \arrow[d, "\rho_{\ell\otimes\mathbb{C}}"] \\ \Lambda(H) \arrow[r, "\otimes\mathbb{C}"'] & \Lambda^{\mathbb{C}}(H \otimes \mathbb{C}). \end{tikzcd} \tag{2.45}$$

Let $E_\pm = \{z \in H \otimes \mathbb{C} | J(z) = \pm\sqrt{-1}z\}$, then

$$H \otimes \mathbb{C} = E_+ \oplus E_-$$

is an orthogonal decomposition of $H \otimes \mathbb{C}$. If $U \in \mathcal{U}_0(H \otimes \mathbb{C})$, then $U(E_\pm) = E_\pm$. Hence we have an isomorphism

$$\mathcal{U}_0(H \otimes \mathbb{C}) \cong \mathcal{U}(E_+) \times \mathcal{U}(E_-),$$

where $\mathcal{U}(E_+)$ denotes the group of unitary operators on E_+, and so on. Also the space $\Lambda^{\mathbb{C}}(H \otimes \mathbb{C})$ is identified with the space of graphs of unitary operators $U \in \mathcal{U}(E_+, E_-)$, $U : E_+ \to E_-$.

Let $\mathfrak{K} : H_J \to E_+, u \mapsto u\otimes 1 - J(u)\otimes\sqrt{-1}$ and $\mathfrak{k} : H_J \to E_-, u \mapsto u\otimes 1 + J(u)\otimes\sqrt{-1}$, be an isomorphism and an anti-isomorphism, respectively, then, the following diagram is commutative.

Lemma 2.63.

$$\begin{tikzcd} H_J \arrow[r, "\tau_\lambda"] \arrow[d, "\mathfrak{K}"'] & H_J \arrow[d, "\mathfrak{k}"] \\ E_+ \arrow[r, "T_\lambda"'] & E_-, \end{tikzcd}$$

where τ_λ is the complex conjugation defined through the identification $H_J \cong \lambda \otimes \mathbb{C}$, and the graph of the unitary operator T_λ is $\lambda \otimes \mathbb{C}$, $\lambda \otimes \mathbb{C} = \{x + T_\lambda(x) | x \in E_+\}$.

Now we have the following proposition.

Proposition 2.64. *Let $\Phi_\lambda : \mathcal{U}_{\mathcal{F}}(H_J) \to \mathcal{F}\Lambda^{\mathbb{C}}_{\lambda\otimes\mathbb{C}}(H \otimes \mathbb{C})$ be a map defined by*

$$\Phi_\lambda(V) = \textit{the graph of the unitary operator} - \mathfrak{k} \circ V \circ \tau_\lambda \circ \mathfrak{K}^{-1}$$

($\in \mathcal{U}(E_+, E_-)$). *Then Φ_λ is an isomorphism and the diagram is commutative*:

$$\begin{tikzcd} \mathcal{F}\Lambda_\lambda(H) \arrow[r, "\otimes\mathbb{C}"] \arrow[dr, "\mathcal{S}_\lambda"'] & \mathcal{F}\Lambda^{\mathbb{C}}_{\lambda\otimes\mathbb{C}}(H\otimes\mathbb{C}) \\ & \mathcal{U}_{\mathcal{F}}(H_J). \arrow[u, "\Phi_\lambda"'] \end{tikzcd} \tag{2.46}$$

Proof. It will be enough to prove the commutativity of the diagram. Let $U \in \mathcal{U}_\lambda(H_J)$. Since $U^{\mathbb{C}}|_{E_\pm}$ can be identified with U through the map $\mathfrak{K}$ and $\mathfrak{k}$, respectively, and we have $U^{\mathbb{C}}(\lambda^\perp\otimes\mathbb{C}) = \{U(x) - U\circ T_\lambda(x)|x \in E_+\} = \{x - U\circ T_\lambda\circ U^{-1}(x)|x \in E_+\}$. By the above lemma $\mathfrak{k}\circ U\circ T_\lambda\circ U^{-1}\circ\mathfrak{K}^{-1} = \mathfrak{k}\circ U\circ\tau_\lambda\circ U^{-1}\circ\tau_\lambda\circ\tau_\lambda\circ\mathfrak{K}^{-1} = \mathfrak{k}\circ U\circ\theta_\lambda(U)\circ\tau_\lambda\circ\mathfrak{K}^{-1}$, which gives the commutativity of the diagram. □

Let W be a closed finite codimensional subspace in $\lambda\otimes\mathbb{C}$ and we denote by $\mathcal{F}\Lambda^{(0)}_W(H\otimes\mathbb{C})$ a subspace of $\mathcal{F}\Lambda^{\mathbb{C}}_{\lambda\otimes\mathbb{C}}(H\otimes\mathbb{C})$ consisting of those subspaces ℓ which do not intersect with W except $\{0\}$. Let $H_W = J(W^\perp\cap(\lambda\otimes\mathbb{C})) + W^\perp\cap(\lambda\otimes\mathbb{C})$, and $\Lambda(H_W)$ be the similar space as $\Lambda(H\otimes\mathbb{C})$ (note that H_W is invariant under the map J). $\Lambda(H_W)$ is identified with the space of unitary operators on $W^\perp\cap(\lambda\otimes\mathbb{C})$. Let $\pi_W : \mathcal{F}\Lambda^{(0)}_W(H\otimes\mathbb{C}) \ni \ell \to (\ell\cap(J(W^\perp\cap\lambda\otimes\mathbb{C}) + \lambda\otimes\mathbb{C}) + W)\cap W^\perp \in \Lambda(H_W)$, and then $\pi_W : \mathcal{F}\Lambda^{(0)}_W(H\otimes\mathbb{C}) \to \Lambda(H_W)$ is a fiber bundle with the contractible fiber. A typical fiber $= \pi_W^{-1}(J(\lambda\otimes\mathbb{C}\cap W^\perp))$ is isomorphic to the space $\hat{\mathcal{B}}(W)\times\mathcal{B}(W, \lambda\otimes\mathbb{C}\cap W^\perp)$, where $\hat{\mathcal{B}}(W)$ is the space of selfadjoint operators on W and $\mathcal{B}(W, W^\perp\cap(\lambda\otimes\mathbb{C}))$ is the space of bounded operators from W to $W^\perp\cap(\lambda\otimes\mathbb{C})$. Unfortunately for any pair of such subspaces W_1 and W_2 satisfying $W_1 \subset W_2$ there are no natural map $\Lambda(H_{W_2}) \to \Lambda(H_{W_1})$ which makes the diagram

$$\begin{array}{ccc} \mathcal{F}\Lambda^{(0)}_{W_2}(H\otimes\mathbb{C}) & \longrightarrow & \mathcal{F}\Lambda^{(0)}_{W_1}(H\otimes\mathbb{C}) \\ \downarrow{\scriptstyle \pi_{W_2}} & & \downarrow{\scriptstyle \pi_{W_1}} \\ \Lambda(H_{W_2}) & \longrightarrow & \Lambda(H_{W_1}) \end{array}$$

commutative. However if we define a map $s_W : \Lambda(H_W) \to \mathcal{F}\Lambda^{(0)}_W(H\otimes\mathbb{C})$ by $s_W(\ell) = \ell + J(W)$, then $\pi_W\circ s_W = \mathrm{Id}$ and we have the following commutative diagram:

$$\begin{array}{ccc} \Lambda(H_{W_2}) & \xrightarrow[\mathbf{i}_{W_1,W_2}]{} & \Lambda(H_{W_1}) \\ \downarrow{\scriptstyle s_{W_2}} & & \downarrow{\scriptstyle s_{W_1}} \\ \mathcal{F}\Lambda^{(0)}_{W_2}(H\otimes\mathbb{C}) & \longrightarrow & \mathcal{F}\Lambda^{(0)}_{W_1}(H\otimes\mathbb{C}), \end{array}$$

where the map $\mathbf{i}_{W_1,W_2} : \Lambda(H_{W_2}) \to \Lambda(H_{W_1})$ is defined by $\mathbf{i}_{W_1,W_2}(\ell) = \ell + J(W_2\cap W_1^\perp)$.

Then for any compact subset K in $\mathcal{F}\Lambda^{\mathbb{C}}_{\lambda\otimes\mathbb{C}}(H\otimes\mathbb{C})$ we can find such a finite codimensional subspace W in $\lambda\otimes\mathbb{C}$ that for any ℓ in K, $\ell\cap W = \{0\}$, so $\bigcup\mathcal{F}\Lambda^{(0)}_W(H\otimes\mathbb{C}) = \mathcal{F}\Lambda^{\mathbb{C}}_{\lambda\otimes\mathbb{C}}(H\otimes\mathbb{C})$. Hence $\text{ind-lim}_{W\to\{0\}}\,\pi_k(\mathcal{F}\Lambda^{(0)}_W(H\otimes\mathbb{C})) = \pi_k(\mathcal{U}_{\mathcal{F}}(H_J)) = \text{ind-lim}_{W\to\{0\}}\,\pi_k(\Lambda(H_W))$. These show that the homotopy groups of $\mathcal{U}_{\mathcal{F}}(H_J)$ coincide with the stable homotopy groups of the unitary group, which together gives the proof of Proposition 2.62, and finally gives us a proof of Theorem 2.54(b) and (c).

3. Maslov index in the infinite dimension

In the last section we proved that the fundamental group of the Fredholm–Lagrangian–Grassmannian is isomorphic to $\mathbb{Z}$. So in this section we define an integer, so called, the *Maslov index*, for arbitrary continuous paths in the Fredholm–Lagrangian–Grassmannian $\mathcal{F}\Lambda_\lambda(H)$. In particular it gives us an explicit isomorphism between the fundamental group of the Fredholm–Lagrangian–Grassmannian and $\mathbb{Z}$. We base on a spectral property of Fredholm operators to define the *Maslov index*, so that our method is valid for both of finite and infinite dimensional cases.

3.1. Maslov index for continuous paths

Let

$$\mathbf{d} : I = [0, 1] \to \mathcal{U}_{\mathcal{F}}(H_J), \qquad t \mapsto \mathbf{d}(t)$$

be a continuous path in $\mathcal{U}_{\mathcal{F}}(H_J)$. First we prove the following lemma.

Lemma 3.1. *There exist a partition* $0 = t_0 < t_1 < \cdots < t_N = 1$ *of the interval* I *and positive numbers* ε_j $(j = 1, \dots , N)$ *with* $0 < \varepsilon_j < \pi$ *such that*

$$e^{\sqrt{-1}(\pi\pm\varepsilon_j)} \in \rho(\mathbf{d}(t)) \tag{3.1}$$

and

$$\sum_{|\theta|\le\varepsilon_j} \dim \operatorname{Ker}(\mathbf{d}(t) - e^{i(\pi+\theta)}) < +\infty \tag{3.2}$$

for $t_{j-1} \le t \le t_j$.

Note here $\rho(\mathbf{d}(t))$ denotes the resolvent set of the operator $\mathbf{d}(t)$.

Proof. Since $\mathbf{d}(t) + \mathrm{Id}$ is a Fredholm operator, for each $t \in I = [0, 1]$ we can find a positive number $\varepsilon_t > 0$ such that

$$\{e^{\sqrt{-1}(\pi+\theta)} | 0 < |\theta| \le \varepsilon_t\} \subset \rho(\mathbf{d}(t)),$$

because -1 is an isolated eigenvalue of $\mathbf{d}(t)$ with finite multiplicity. So there exist positive numbers $\delta_t^{\pm} > 0$ such that the projection operator P_s defined by

$$P_s = \frac{1}{2\pi\sqrt{-1}} \int_{|u+1|=\varepsilon_t} (u - \mathbf{d}(s))^{-1}\, du, \tag{3.3}$$

has the constant rank equal to $\dim \mathrm{Ker}(\mathbf{d}(t)+\mathrm{Id})$ for $s \in [t-\delta_t^-, t+\delta_t^+]$, because $\{P_s\}$ is a norm continuous family:

$$\dim P_s(H_J) = \dim \mathrm{Ker}(\mathbf{d}(t)+\mathrm{Id}), \quad s \in [t-\delta_t^-, t+\delta_t^+]. \tag{3.4}$$

Note that the continuity of the family of projection operators $\{P_s\}$ is proved by using the "resolvent equation":

$$(u-\mathbf{d}(s))^{-1} - (u-\mathbf{d}(t))^{-1} = (u-\mathbf{d}(s))^{-1}(\mathbf{d}(s)-\mathbf{d}(t))(u-\mathbf{d}(t))^{-1}.$$

The continuity of $\{\mathbf{d}(t)\}$ is reflected by this equation to the continuity of the family $\{P_s\}$.

Hence we have an open covering $\{(t-\delta_t^-, t+\delta_t^+)\}_{t\in I}$ of the interval I and positive numbers $\{\varepsilon_t\}_{t\in I}$ such that for $s \in [t-\delta_t^-, t+\delta_t^+]$

$$\sum_{|\theta|\le\varepsilon_t} \dim \mathrm{Ker}(\mathbf{d}(s) - \mathrm{e}^{\sqrt{-1}(\pi+\theta)}) = \dim \mathrm{Ker}(\mathbf{d}(t)+\mathrm{Id}),$$

$$\mathrm{e}^{\sqrt{-1}(\pi\pm\varepsilon_t)} \in \rho(\mathbf{d}(s)).$$

Now we can choose an enough number of points $\{s_i\}_{i=0}^{N-1}$ satisfying following properties:

$$0 = s_0 < s_1 < \cdots < s_{N-1} = 1 \text{ such that,}$$

$$s_{i-1} < s_i - \delta_{s_i}^-,$$

$$s_{i-1} + \delta_{s_{i-1}}^+ < s_i,$$

$$s_i - \delta_{s_i}^- < s_{i-1} + \delta_{s_{i-1}}^+.$$

Here if necessary, we replace $\delta_{s_i}^\pm$ by a smaller one (but then the number of the points $\{s_i\}$ will increase). Finally we define the point t_k ($k = 0, \ldots, N$) in such a way that

$$t_0 = s_0 = 0, \quad t_1 = \delta_0^+, \quad t_2 = s_1 + \delta_{s_1}^+, \quad t_3 = s_2 + \delta_{s_2}^+, \ldots,$$

$$t_{N-1} = s_{N-2} + \delta_{s_{N-2}}^+, \quad t_N = s_{N-1} = 1,$$

and on the each interval $[t_{k-1}, t_k]$ we take the positive number $\varepsilon_k = \varepsilon_{s_k}$. Then we have a desired partition of the interval I and positive numbers satisfying (3.1) and (3.2). □

We now define a quantity, we call it "*unitary Maslov index*", and denote by $\mathbf{M}(\{\mathbf{d}(t)\})$ of a continuous curve $\{\mathbf{d}(t)\}_{t\in I} \subset \mathcal{U}_{\mathcal{F}}(H_J)$.

Definition 3.2. Let $\{t_j\}_{j=0}^N$ be the partition of the interval I and $\{\varepsilon\}_{j=1}^N$ positive numbers satisfying (3.1) and (3.2) as in the above lemma, then we define

$$\mathbf{M}(\{\mathbf{d}(t)\}) = \sum_{j=1}^{N} (k(t_j, \varepsilon_j) - k(t_{j-1}, \varepsilon_j)) \tag{3.5}$$

with

$$k(t,\epsilon_j)=\sum_{0\le\theta\le\varepsilon_j}\dim\mathrm{Ker}(\mathbf{d}(t)-\mathrm{e}^{\mathrm{i}(\pi+\theta)})$$

for $t_{j-1}\le t\le t_j$.

In order that the definition has a meaning, we need to prove the following proposition.

Proposition 3.3. *The definition of the quantity* $\mathbf{M}(\{\mathbf{d}(t)\})$ *depend neither on the choices of the partition* $\{t_j\}_{j=0}^N$ *of the interval I nor on the positive numbers* $\{\varepsilon_j\}_{j=1}^N$ *satisfying* (3.1) and (3.2).

This follows from the following lemma: let $\{t_j\}_{j=0}^N$ be the partition of the interval I and $\{\tilde\varepsilon_j\}_{j=1}^N$ another positive numbers satisfying (3.1) and (3.2).

Lemma 3.4. *The two integers coincide defined in* (3.5) *one in terms of the partition* $\{t_j\}_{j=0}^N$ *and positive numbers* $\{\varepsilon_j\}_{j=1}^N$ *and other in terms of the "same partition"* $\{t_j\}_{j=0}^N$ *and "different positive numbers"* $\{\tilde\varepsilon_j\}_{j=1}^N$.

Proof. Since both of $\mathrm{e}^{\sqrt{-1}(\pi+\varepsilon_j)}$ and $\mathrm{e}^{\sqrt{-1}(\pi+\tilde\varepsilon_j)}\in\rho(\mathbf{c}(t))$ on each small interval $[t_{j-1},t_j]$, the difference of the dimensions $k(t,\varepsilon_j)-k(t,\tilde\varepsilon_j)$ is constant on the interval $[t_{j-1},t_j]$. Hence we have

$$k(t_j,\varepsilon_j)-k(t_{j-1},\varepsilon_j)=k(t_j,\tilde\varepsilon_j)-k(t_{j-1},\tilde\varepsilon_j),$$

which proves the lemma. □

Proof of Proposition of 3.3. By adding a suitable number of points both in the partitions $\{t_j\}$ and $\{\tilde t_l\}$, we may assume that $t_{j-1}<\tilde t_j<t_j$ for each j. Then from Lemma 3.4 we have

$$k(t_j,\varepsilon_j)-k(t_{j-1},\varepsilon_j), \tag{3.6}$$

$$=k(t_j,\varepsilon_j)-k(\tilde t_j,\varepsilon_j)+k(\tilde t_j,\varepsilon_j)-k(t_{j-1},\varepsilon_j), \tag{3.7}$$

$$=k(t_j,\tilde\varepsilon_{j+1})-k(\tilde t_j,\tilde\varepsilon_{j+1})+k(\tilde t_j,\tilde\varepsilon_j)-k(t_{j-1},\tilde\varepsilon_j), \tag{3.8}$$

which gives us the coincidence of the two integers by adding (3.6) and (3.8) with respect to j. □

Notation 3.5.

(a) Let $\{\mathbf{d}_1(t)\}_{t\in[0,1]}$ and $\{\mathbf{d}_2(t)\}_{t\in[0,1]}$ be continuous curves with the relation $\mathbf{d}_1(1)=\mathbf{d}_2(0)$, then we denote the catenation of these two curves by $\mathbf{d}_1*\mathbf{d}_2$:

$$(\mathbf{d}_1*\mathbf{d}_2)(t)=\begin{cases}\mathbf{d}_1(\frac{1}{2}t) & \text{for } 0\le t\le\frac{1}{2},\\ \mathbf{d}_2(2t-1) & \text{for } \frac{1}{2}\le t\le 1.\end{cases}$$

(b) The curve $-\mathbf{d}$ denotes the curve defined by $-\mathbf{d}(t) = \mathbf{d}(1-t)$, $t \in I$.

This unitary Maslov index has the following properties.

Theorem 3.6.

(a) *Additivity under the catenation of the paths, and*
(b) *Modulo sign and additive constants, it is only a homotopy invariant of curves in $\mathcal{U}_{\mathcal{F}}(H_J)$ with fixed endpoints and distinguishes the homotopy classes.*

Proof. (a) The additivity follows from the very definition of the quantity $\mathbf{M}\{\mathbf{d}(t)\}$.

(b) Let $\{\mathbf{w}(s,t)\}_{(s,t)\in I\times I} \subset \mathcal{U}_{\mathcal{F}}(H_J)$ be a continuous two-parameter family. By the similar continuity arguments for the projection operator (3.3) in the proof of Lemma 3.1, for each $s \in I$ there are a positive number $c_s > 0$, the partition $\{t_j\}$ of the interval and the positive numbers $\{\varepsilon_j\}$ such that (3.1) and (3.2) hold for $t_{j-1} \le t \le t_j$ and $|s'-s| \le c_s$:

$$\mathrm{e}^{\sqrt{-1}(\pi\pm\varepsilon_j)} \in \rho(\mathbf{w}(s',t))$$

and

$$\sum_{|\theta|\le\varepsilon_j} \dim \operatorname{Ker}(\mathbf{w}(s',t) - \mathrm{e}^{\mathrm{i}(\pi+\theta)}) < \infty.$$

So on the each small rectangle $[t_{j-1}, t_j] \times [s, s+c_s]$, $v \in [s, s+c_s]$

$$\begin{aligned}
&\sum_{0\le\theta\le\varepsilon_j} \dim \operatorname{Ker}(\mathbf{w}(s+v,t_j) - \mathrm{e}^{\mathrm{i}(\pi+\theta)}) - \sum_{0\le\theta\le\varepsilon_j} \dim \operatorname{Ker}(\mathbf{w}(s+v,t_{j-1}) - \mathrm{e}^{\mathrm{i}(\pi+\theta)})\\
&\quad + \sum_{0\le\theta\le\varepsilon_j} \dim \operatorname{Ker}(\mathbf{w}(s+v,t_{j-1}) - \mathrm{e}^{\mathrm{i}(\pi+\theta)}) - \sum_{0\le\theta\le\varepsilon_j} \dim \operatorname{Ker}(\mathbf{w}(s,t_{j-1}) - \mathrm{e}^{\mathrm{i}(\pi+\theta)})\\
&\quad = \sum_{0\le\theta\le\varepsilon_j} \dim \operatorname{Ker}(\mathbf{w}(s+v,t_j) - \mathrm{e}^{\mathrm{i}(\pi+\theta)}) - \sum_{0\le\theta\le\varepsilon_j} \dim \operatorname{Ker}(\mathbf{w}(s,t_j) - \mathrm{e}^{\mathrm{i}(\pi+\theta)})\\
&\quad + \sum_{0\le\theta\le\varepsilon_j} \dim \operatorname{Ker}(\mathbf{w}(s,t_j) - \mathrm{e}^{\mathrm{i}(\pi+\theta)}) - \sum_{0\le\theta\le\varepsilon_j} \dim \operatorname{Ker}(\mathbf{w}(s,t_{j-1}) - \mathrm{e}^{\mathrm{i}(\pi+\theta)}).
\end{aligned}$$

By adding above equalities with respect to j we have in general (locally with respect to the parameter s)

$$\begin{aligned}
&\mathbf{M}(\{\mathbf{w}(s+v,0)\}_{0\le v\le c_s}) + \mathbf{M}(\{\mathbf{w}(s+c_s,t)\}_{t\in I})\\
&\quad = \mathbf{M}(\{\mathbf{w}(s,t)\}_{t\in I}) + \mathbf{M}(\{\mathbf{w}(s+v,1)\}_{0\le v\le c_s}),
\end{aligned}$$

and then on the rectangle $I \times I$

$$\mathbf{M}(\{\mathbf{w}(s,0)\}_{s\in I}) + \mathbf{M}(\{\mathbf{w}(1,t)\}_{t\in I}) = \mathbf{M}(\{\mathbf{w}(0,t)\}_{t\in I}) + \mathbf{M}(\{\mathbf{w}(s,1)\}_{s\in I}).$$

Now here we assume that $\mathbf{w}(s,0) \equiv \mathbf{w}(0,0)$ and $\mathbf{w}(s,1) \equiv \mathbf{w}(0,1)$ ($s \in I$), hence $\mathbf{M}(\{\mathbf{w}(0,t)\}_{t\in I}) = \mathbf{M}(\{\mathbf{w}(1,t)\}_{t\in I})$, and this shows the homotopy invariance of the integer $\mathbf{M}(\{\mathbf{w}(t)\})$.

The uniqueness (mod additive constant and signature) follows from the fact that $\pi_1(\mathcal{U}_{\mathcal{F}}(H_J)) \cong \mathbb{Z}$. □

The space $U_{\mathcal{F}}(H_J)$ is closed under the adjoint operation, so we have the following proposition.

Proposition 3.7. $\mathbf{M}(\{\mathbf{w}(t)\}) = -\mathbf{M}(\{\mathbf{w}(t)^*\})$.

Using this "unitary Maslov index" we give a functional analytic definition of the infinite version of the *Maslov index* for arbitrary continuous paths in the Fredholm–Lagrangian–Grassmannian.

Let $\mu : I \to \mathcal{F}\Lambda_\lambda(H)$ be a continuous path in $\mathcal{F}\Lambda_\lambda(H)$ (so that $\mathcal{S}_\lambda \circ \mu$ is a continuous path in $\mathcal{U}_{\mathcal{F}}(H_J)$).

Definition 3.8. We define the Maslov index of the curve $\{\mu(t)\}$ with respect to λ by

$$\mathbf{Mas}(\{\mu(t)\}, \lambda) = \mathbf{M}(\{\mathcal{S}_\lambda(\mu(t))\}).$$

By Theorem 2.54, the Maslov index inherits the all properties of the "unitary Maslov index".

In the case that $\mathcal{F}\Lambda_\lambda(H) = \mathcal{F}\Lambda_\mu(H)$, but $\lambda \neq \mu$, then Maslov cycles $\mathfrak{M}_\lambda(H)$ and $\mathfrak{M}_\mu(H)$ do not coincide. Hence Maslov indexes for a path with respect to $\mathfrak{M}_\lambda(H)$ and $\mathfrak{M}_\mu(H)$ will not coincide in general. However for loops, as in the finite dimensional case we have the following proposition.

Proposition 3.9. *Let* $\lambda, \mu \in \Lambda(H)$ *and assume that* $\mu = U_1(\lambda)$ *with a unitary operator* $U_1 \in \mathcal{U}_{\mathrm{res}}(H_J)$. *Then for any continuous loops* $\{\mathbf{c}(t)\}_{t\in[0,1]}$ *in* $\mathcal{F}\Lambda_\lambda(H) = \mathcal{F}\Lambda_\mu(H)$ *their Maslov indexes coincide*:

$$\mathbf{Mas}(\{\mathbf{c}(t)\}, \lambda) = \mathbf{Mas}(\{\mathbf{c}(t)\}, \mu).$$

Proof. Let $\{U_s\}_{s\in[0,1]}$ be a continuous curve in $\mathcal{U}_{\mathrm{res}}(H_J)$ which joins λ and μ, that is, $U_0 = \mathrm{Id}$ and $U_1(\lambda) = \mu$. Note then for each $s \in [0, 1]$, $\mathcal{F}\Lambda_{U_s(\lambda)}(H) = \mathcal{F}\Lambda_\lambda(H)$.

We define a map $\mathbf{h} : I \times I \to \mathcal{U}_{\mathcal{F}}(H_J)$:

$$\mathbf{h}(s, t) = \begin{cases} \mathcal{S}_{U_{2ts}(\lambda)}(\mathbf{c}(t)) & \text{for } (s, t) \in [0, 1] \times [0, \frac{1}{2}], \\ \mathcal{S}_{U_{(2-2t)s}(\lambda)}(\mathbf{c}(t)) & \text{for } (s, t) \in [0, 1] \times [\frac{1}{2}, 1]. \end{cases}$$

Then $\{\mathbf{h}(s, t)\}$ is a homotopy between the loop $\{\mathcal{S}_\lambda(\mathbf{c}(t))\}$ and the loop $\{\mathbf{h}(1, t)\}$ with the fixed common initial and end point $\mathcal{S}_\lambda(\mathbf{c}(0)) = \mathcal{S}_\lambda(\mathbf{c}(1)) = \mathbf{h}(s, 0) = \mathbf{h}(s, 1)$, $s \in [0, 1]$. Hence

$$\mathbf{Mas}(\{\mathbf{c}(t)\}, \lambda) = \mathbf{M}(\{\mathcal{S}_\lambda(\mathbf{c}(t))\}) = \mathbf{M}(\{\mathbf{h}(1, t)\}).$$

By the same way for the loops $\{\mathbf{h}(1, t)\}$ and $\mathcal{S}_\mu(\mathbf{c}(t))$ we can construct a homotopy in $\mathcal{U}_{\mathcal{F}}(H_J)$ between them and these show the coincidence of the two Maslov indexes. □

Corollary 3.10. *Let* $\{\mathbf{c}(t)\}_{t\in[0,1]}$ *be a continuous path in* $\mathcal{F}\Lambda_\lambda(H)$ *such that* $\mathbf{c}(0), \mathbf{c}(1) \notin \mathfrak{M}_\lambda$ *and let* $\{U_s\}_{s\in[0,1]} \subset \mathcal{U}_{\mathrm{res}}(H_J)$ *be a continuous family with* $U_0 = \mathrm{Id}$. *We assume that* $\mathbf{c}(0), \mathbf{c}(1) \notin \mathfrak{M}_{U_s(\lambda)}(H)$ *for all* $s \in [0, 1]$, *then for all s*

$$\mathbf{Mas}(\{\mathbf{c}(t)\}, \lambda) = \mathbf{Mas}(\{\mathbf{c}(t)\}, U_s(\lambda)).$$

3.2. Hörmander index in the infinite dimension

Let $\lambda, \mu \in \Lambda(H)$ and assume that $\mu = U(\lambda)$ with a unitary operator U of the form Id + compact operator, and let $\{\mathbf{c}(t)\}_{t\in[0,]}$ be a continuous curve in $\mathcal{F}\Lambda_\lambda(H) = \mathcal{F}\Lambda_\mu(H)$.

Proposition 3.11. *The difference*

$$\mathbf{Mas}(\{\mathbf{c}(t)\}, \lambda) - \mathbf{Mas}(\{\mathbf{c}(t)\}, \mu)$$

depends only on the end points.

Proof. Let $\{\tilde{\mathbf{c}}(t)\}$ be another path with $\tilde{\mathbf{c}}(0) = \mathbf{c}(0)$, $\tilde{\mathbf{c}}(1) = \mathbf{c}(1)$, then

$$\begin{aligned}\mathbf{Mas}(\{\mathbf{c} * (-\tilde{\mathbf{c}})(t)\}, \lambda) &= \mathbf{Mas}(\{\mathbf{c}(t)\}, \lambda) - \mathbf{Mas}(\{\tilde{\mathbf{c}}(t)\}, \mu) = \mathbf{Mas}(\{\mathbf{c} * (-\tilde{\mathbf{c}})(t)\}, \mu) \\ &= \mathbf{Mas}(\{\mathbf{c}(t)\}, \mu) - \mathbf{Mas}(\{\tilde{\mathbf{c}}(t)\}, \mu)\end{aligned}$$

by Proposition 3.9. Hence we have the desired result. □

Using this property we can define an infinite dimensional version of an integer, called "*Hörmander index*", for four Lagrangian subspaces.

Definition 3.12. Let $\mu = U(\lambda) \in \Lambda(H)$ with $U = \mathrm{Id} + \text{compact operator}$, and let ℓ_0 and ℓ_1 be in $\mathcal{F}\Lambda_\lambda(H) = \mathcal{F}\Lambda_\mu(H)$. Also let $\{\mathbf{c}(t)\}$ be a curve in $\mathcal{F}\Lambda_\lambda(H) = \mathcal{F}\Lambda_\mu(H)$ joining ℓ_0 and ℓ_1. Then we call the difference

$$\mathbf{Mas}(\{\mathbf{c}(t)\}, \lambda) - \mathbf{Mas}(\{\mathbf{c}(t)\}, \mu)$$

the *Hörmander index* in the infinite dimension and denote it by

$$\sigma(\ell_0, \ell_1; \lambda, \mu). \tag{3.9}$$

Let $\mu = U(\lambda)$ be as above and let $\ell_0, \ell_1, \ell_2 \in \mathcal{F}\Lambda_\lambda(H) = \mathcal{F}\Lambda_\mu(H)$, then the Hörmander index $\sigma(\ell_0, \ell_1; \lambda, \mu)$ has the following properties.

Proposition 3.13.

$$\sigma(\ell_0, \ell_1; \lambda, \mu) = -\sigma(\ell_1, \ell_0; \lambda, \mu), \tag{3.10}$$

$$\sigma(\ell_0, \ell_1; \lambda, \mu) = -\sigma(\ell_0, \ell_1; \mu, \lambda), \tag{3.11}$$

$$\sigma(\ell_0, \ell_1; \lambda, \mu) + \sigma(\ell_1, \ell_2; \lambda, \mu) = \sigma(\ell_0, \ell_2; \lambda, \mu). \tag{3.12}$$

Let $\nu = W(\lambda)$ also with a unitary operator $W = \mathrm{Id} +$ compact operator, then the cocycle condition with respect to the last two components hold:

$$\sigma(\ell_0, \ell_1; \lambda, \mu) + \sigma(\ell_0, \ell_1; \mu, \nu) = \sigma(\ell_0, \ell_1; \lambda, \nu). \tag{3.13}$$

If we assume moreover $\ell_1 = V(\ell_0)$ with a unitary operator V of the form $\mathrm{Id}+$compact operator, then

$$\sigma(\ell_0, \ell_1; \lambda, \mu) = -\sigma(\lambda, \mu; \ell_0, \ell_1). \tag{3.14}$$

Proof. Four properties except last one follow directly from the definition itself.

Let $\{U_s\}_{s\in[0,1]}$ and $\{V_t\}_{t\in[0,1]}$ be such curves of unitary operators that each operator U_s and V_t are of the form of $\mathrm{Id} +$ compact operator and assume $U_0 = \mathrm{Id}$, $U_1(\lambda) = \mu$, $V_0 = \mathrm{Id}$ and $V_1(\ell_0) = \ell_1$. Then for any $s \in [0, 1]$, $\mathcal{F}\Lambda_{U_s(\lambda)}(H) = \mathcal{F}\Lambda_\lambda(H)$, and for any (s, t) $(U_s(\lambda), V_t(\ell_0))$ is a Fredholm pair. So the two-parameter continuous family of unitary operators $\{\mathcal{S}_{U_s(\lambda)}(V_t(\ell_0))\}$ are in $\mathcal{U}_{\mathcal{F}}(H_J)$. Let us define a curve $\{\mathbf{c}(t)\}_{0\le t\le 4}$:

$$\mathbf{c}(t) = \begin{cases} \mathcal{S}_\lambda(V_t(\ell_0)) & \text{for } 0 \le t \le 1, \\ \mathcal{S}_{U_{t-1}(\lambda)}(\ell_1) & \text{for } 1 \le t \le 2, \\ \mathcal{S}_\mu(V_{3-t}(\ell_0)) & \text{for } 2 \le t \le 3, \\ \mathcal{S}_{U_{4-t}(\lambda)}(\ell_0) & \text{for } 3 \le t \le 4. \end{cases}$$

The unitary Maslov index $\mathbf{M}(\{\mathbf{c}(t)\}_{0\le t\le 4})$ of this curve is zero, so

$$\begin{aligned}&\mathbf{M}(\{\mathcal{S}_\lambda(V_t(\ell_0))\}_{t\in[0,1]}) - \mathbf{M}(\{\mathcal{S}_\mu(V_t(\ell_0))\}_{t\in[0,1]}) \\ &\quad = \mathbf{M}(\{\mathcal{S}_{U_t(\lambda)}(\ell_0)\}_{t\in[0,1]}) - \mathbf{M}(\{\mathcal{S}_{U_t(\lambda)}(\ell_1)\}_{t\in[0,1]}),\end{aligned}$$

and by Proposition 2.49 this equal to

$$= -\mathbf{M}(\{\mathcal{S}_{\ell_0}(U_t(\lambda))\}_{t\in[0,1]}) + \mathbf{M}(\{\mathcal{S}_{\ell_1}(U_t(\lambda))\}_{t\in[0,1]}).$$

Hence

$$\sigma(\ell_0, \ell_1; \lambda, \mu) = -\sigma(\lambda, \mu; \ell_0, \ell_1). \qquad \square$$

Remark 3.14. The Hörmander index was first introduced in the paper [20] to describe the phase transitions in the oscillatory integral representation of Fourier integral operators or Lagrangian distributions for the global formulation in terms of, so called, Maslov line bundle. It was given also as a Čech cocycle. Our definition above is given in terms of the Maslov index, and we need not to assume the transversality conditions between each of the first two Lagrangian subspaces and each of the last two Lagrangian subspaces. The reason is, of course, the Maslov index is defined for not only loops but also any paths. In earlier papers written before the papers [4,16,29] it was only considered for loops or with the assumption that the end points of paths do not meet with a particularly fixed Maslov cycle. However in order to construct the Maslov line bundle, it is enough to consider the indexes for four Lagrangian subspaces satisfying transversality conditions.

In the next subsection we construct an infinite dimensional analogue of the Maslov line bundle which will be turn out to be a kind of the universal Maslov line bundle.

3.3. Universal covering space of the Fredholm–Lagrangian–Grassmannian

In this section we characterize the universal covering space $\widetilde{\mathcal{F}\Lambda}_\ell(H)$ of the Fredholm–Lagrangian–Grassmannian $\mathcal{F}\Lambda_\ell(H)$ in terms of the Hörmander index. We show the transition functions of the principal bundle $\pi : \widetilde{\mathcal{F}\Lambda}_\ell(H) \to \mathcal{F}\Lambda_\ell(H)$ are given by the Hörmansder index. Here we understand the space $\widetilde{\mathcal{F}\Lambda}_\ell(H)$ consisting of pairs $([\mathbf{c}], \mathbf{c}(1))$ of homotopy classes $[\mathbf{c}]$ of paths $\{\mathbf{c}(t)\}$ in $\mathcal{F}\Lambda_\ell(H)$ with the common initial point $\mathbf{c}(0) = \ell^\perp$ and its end point $\mathbf{c}(1)$.

Let $\lambda \in \Lambda(H)$ and assume $\lambda \sim \ell$, and we define a map

$$\phi_\lambda : \mathcal{F}\Lambda_\ell(H) \times \mathbb{Z} \to \widetilde{\mathcal{F}\Lambda}_\ell(H)$$

by

$$\phi_\lambda : (\theta, n) \mapsto [\mathbf{c}(t)],$$

where $\{\mathbf{c}(t)\}$ is a path joining $\ell^\perp$ and θ, and $\mathrm{Mas}(\{\mathbf{c}(t)\}, \ell) = n$. Note that we know the homotopy class of such paths is uniquely determined.

By the definition of the topology on the space $\widetilde{\mathcal{F}\Lambda}_\ell(H)$, it is immediate to show that the map is bijective, and not continuous on the whole space of definition.

Proposition 3.15. *The map ϕ_λ restricted to the open subset*

$$(\mathcal{F}\Lambda_\ell(H) \setminus \mathfrak{M}_\lambda(H)) \times \mathbb{Z} = \mathbf{O}_\lambda \times \mathbb{Z}$$

is an isomorphism with the space

$$\pi^{-1}(\mathcal{F}\Lambda_\ell(H) \setminus \mathfrak{M}_\lambda(H)).$$

Now let $\lambda \sim \ell$, $\mu \sim \ell$ and let $\nu \in \mathbf{O}_\lambda \cap \mathbf{O}_\mu$. Then if $\phi_\lambda(\nu, n) = \phi_\mu(\nu, m)$, then $n - m = \sigma(\ell^\perp, \nu; \lambda, \mu)$ and so by the cocycle condition (3.13) we have the following proposition.

Proposition 3.16. *The maps*

$$g_{\lambda,\mu} : \mathbf{O}_\lambda \cap \mathbf{O}_\mu \to \mathbb{Z}, \qquad \nu \mapsto \sigma(\ell^\perp, \nu; \lambda, \mu) \tag{3.15}$$

are the transition functions of the principal bundle $\pi : \widetilde{\mathcal{F}\Lambda}_\ell(H) \to \mathcal{F}\Lambda_\ell(H)$ with the structure group $\pi_1(\mathcal{F}\Lambda_\ell(H)) \cong \mathbb{Z}$.

From this fact we can define the following definition.

Definition 3.17. We call the complex line bundle $\mathcal{L}_\ell$ on $\mathcal{F}\Lambda_\ell(H)$ defined by the transition functions $\{h_{\lambda,\mu}\}$ $(\lambda, \mu \sim \ell)$

$$h_{\lambda,\mu}(\nu) = \mathrm{e}^{\sqrt{-1}(\pi/2)\sigma(\ell^\perp, \nu; \lambda, \mu)}$$

the universal Maslov line bundle.

In fact, we have the following: let $H = H_0 + H_1$ be an orthogonal direct sum by symplectic subspaces with $\dim H_0 = 2n < +\infty$, and we fix Lagrangian subspaces $\ell_0 \in H_0$ and ℓ_1 in H_1. Then we have an embedding $\mathbf{i} : \Lambda(H_0) \to \Lambda(H)$

$$\mathbf{i} : \Lambda(H_0) \to \mathcal{F}\Lambda_{\ell_0 \oplus \ell_1}(H), \qquad \mathbf{i} : \theta \mapsto \theta \oplus \ell^{\perp}. \tag{3.16}$$

Then the map $\mathbf{i}$ gives a relation between the Hörmander indexes on $\Lambda(H_0)$ and $\mathcal{F}\Lambda_{(\ell_0 \oplus \ell_1)}(H)$: for $\lambda, \mu \in \Lambda(H_0)$

$$\sigma(\ell_0^{\perp}, \theta; \lambda, \mu) = \sigma((\ell_0 \oplus \ell_1)^{\perp}, \theta \oplus \ell_1^{\perp}; \lambda \oplus \ell_1, \mu \oplus \ell_1).$$

Hence we have the following proposition.

Proposition 3.18. $\mathbf{i}^*(\mathcal{L}_{\ell_0 \oplus \ell_1}) \cong$ *the Maslov line bundle on* $\Lambda(H_0)$.

Remark 3.19. The collections of the vector spaces

$$\coprod_{\mu \in \mathcal{F}\Lambda_\lambda(H)} \lambda \cap \mu$$

and

$$\coprod_{\mu \in \mathcal{F}\Lambda_\lambda(H)} H/(\lambda + \mu)$$

are not apparently vector bundles, but

$$\coprod_{\mu \in \mathcal{F}\Lambda_\lambda(H)} \left(\bigwedge^{\max} \lambda \cap \mu\right)^* \otimes \left(\bigwedge^{\max} H/(\lambda + \mu)\right)$$

has a line bundle structure. Here $\bigwedge^{\max} \lambda \cap \mu$ means the highest degree exterior product. This is isomorphic with the induced bundle of the Quillen determinant line bundle on the space of all Fredholm operators by the map $\mu \mapsto \mathcal{P}_\lambda + \mathcal{P}_\mu$ and also its complexification is isomorphic with the induced bundle by the map $\mathrm{Id} + \mathcal{S}_\lambda$ [15]. This is a trivial line bundle.

3.4. Bilinear forms and Maslov index

For “differentiable curves” in $\mathcal{U}_{\mathcal{F}}(H_J)$ satisfying a certain non-degeneracy condition, there is another way of describing the “unitary Maslov index”. We define a symmetric bilinear form which is analogous to Duistermaat [11] and Robbin–Salamon [29].

Let $\{\mathbf{c}(t)\}$ be a “C^1-path” in $\mathcal{U}_{\mathcal{F}}(H_J)$. Here we mean C^1-path in the following sense: there is a continuous family $\{D_t\}_{t \in I}$ of bounded operators $D_t \in \mathcal{B}(H_J)$ satisfying

$$\left\| \frac{1}{\delta}(\mathbf{c}(t+\delta) - \mathbf{c}(t)) - t \cdot D_t \right\| = o(1) \tag{3.17}$$

on the interval I. We denote $D_t = (\mathrm{d}/\mathrm{d}t)\mathbf{c}(t) = \dot{\mathbf{c}}(t)$.

Definition 3.20.

(a) A parameter t^* with $0 \le t^* \le 1$ is called a *crossing* for the family $\{\mathbf{c}(t)\}$, if $\mathrm{Ker}(\mathbf{c}(t^*) + \mathrm{Id}) \neq \{0\}$.

(b) We define the *crossing form* at a crossing t^* as a symmetric bilinear form on $\mathrm{Ker}(\mathbf{c}(t^*)+\mathrm{Id})$ by

$$\tilde{Q}_{\mathfrak{M}}(x, y) = \frac{\mathrm{d}}{\mathrm{d}t}\langle x, R_t(y)\rangle_J\bigg|_{t=t^*} \qquad \text{for } x, y \in \mathrm{Ker}(\mathbf{c}(t^*) + \mathrm{Id}),$$

where $\{R_t\}$ is a family of bounded selfadjoint operators given by the relation $\mathbf{c}(t) = \mathbf{c}(t^*)\,\mathrm{e}^{\sqrt{-1}R_t}$, $R_{t^*} = 0$, i.e.

$$R_t = -\sqrt{-1}\log(\mathbf{c}(t^*)^{-1} \circ \mathbf{c}(t)) \quad (\text{for “log” see Remark 2.18 below}).$$

Then $\dot{\mathbf{c}}(t^*) = \sqrt{-1}\mathbf{c}(t^*) \circ \dot{R}_{t^*}$.

(c) We call a crossing t^* is *regular*, if the form $\tilde{Q}_{\mathfrak{M}}$ is non-degenerate on $\mathrm{Ker}(\mathbf{c}(t^*) + \mathrm{Id})$.

Remark 3.21. The logarithm above is defined by the integral

$$\log M = \int_{-\infty}^{0} \{(u - M)^{-1} - (u - 1)^{-1}\,\mathrm{Id}\}\,\mathrm{d}u \tag{3.18}$$

for a bounded invertible operator $M \in \mathcal{B}(H_J)$ whose spectrum $\sigma(M)$ does not contain any negative real numbers: $\sigma(M) \cap (-\infty, 0] = \phi$.

The integral converges in the operator norm and the resulting family is again C^1-class, if $\{M(t)\}$ is so. The derivative in the sense of (3.17) is given by the integral:

$$\frac{\mathrm{d}}{\mathrm{d}t}\log M(t) = \int_{-\infty}^{0}\left\{(u - M(t))^{-1} \circ \frac{\mathrm{d}}{\mathrm{d}t}M(t) \circ (u - 1)^{-1}\right\}\mathrm{d}u. \tag{3.19}$$

For our case $M(t) = \mathbf{c}(t^*)^{-1} \circ \mathbf{c}(t)$, $|t - t^*| \ll 1$, by a direct calculation we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\log \mathbf{c}(t^*)^{-1} \circ \mathbf{c}(t)_{|t=t^*}\\ &\quad = -\sqrt{-1}\int_{-\infty}^{0}\left\{(u - \mathrm{Id})^{-2} \circ \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{c}(t^*)^{-1} \circ \mathbf{c}(t))_{|t=t^*}\right\}\mathrm{d}u = \dot{R}_{t^*}.\end{aligned}$$

Proposition 3.22. *Let $\{\mathbf{c}(t)\}$ be a path in $\mathcal{U}_{\mathcal{F}}(H_J)$ of class C^1 and $0 < t^* < 1$ a regular crossing. Then there exists a real $\delta > 0$ such that*

$$\mathbf{M}(\{\mathbf{c}(t)\}_{|t-t^*|\le\delta}) = \mathrm{sign}\,\tilde{Q}_{\mathfrak{M}}.$$

Before proving this proposition we give a lemma which describes a behavior of eigenvalues close to zero of a family of selfadjoint Fredholm operators under a certain non-degeneracy condition (see [22]).

Lemma 3.23. *Let $\{A_t\}_{|t|\ll 1}$ be a C^1-class family of selfadjoint Fredholm operators on a Hilbert space H. Assume that the symmetric bilinear form on* $\mathrm{Ker}\,A_0$

$$Q(x, y) = \frac{\mathrm{d}}{\mathrm{d}t}(x, A_t(y))_{|t=0} = (x, \dot{A}_0(y)), \quad x, y \in \mathrm{Ker}\,A_0$$

is non-degenerate. Then there exists a positive number δ *such that for* $0 < t \leq \delta$ *there exist* p *positive eigenvalues and* q *negative eigenvalues of the operator* A_t, *where* $p - q = \operatorname{sign} Q$, $p + q = \dim \operatorname{Ker} A_0$. *Also for* $-\delta \leq t < 0$ *the opposite situations hold.*

Proof. From the Fredholmness assumption of the continuous family $\{A_t\}$ there exist positive numbers δ and ε such that the projection operators P_t for $|t| \leq \delta$ defined by

$$P_t = \frac{1}{2\pi\sqrt{-1}} \int_{|u|=\varepsilon} (u - A_t)^{-1}\,\mathrm{d}u \tag{3.20}$$

have the constant rank equal to $\dim \operatorname{Ker} A_0$, and the range of each $P_t = \sum_{|u|<\varepsilon} \operatorname{Ker}(A_t - u)$. By the approximation arguments we know that the bilinear forms

$$\left(\frac{1}{t} \cdot A_t \circ P_t(x),\, P_t(y)\right), \quad x, y \in \operatorname{Ker} A_0$$

and

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} A_{t|t=0}(x),\, y\right), \quad x, y \in \operatorname{Ker} A_0$$

are uniformly close. In fact, for $x, y \in \operatorname{Ker} A_0$

$$\begin{aligned}
&\left(\frac{1}{t} \cdot A_t \circ P_t(x),\, P_t(y)\right) - (\dot{A}_0(x), y)\\
&\quad = \left(\left(\frac{1}{t} \cdot (A_t - A_0) - \dot{A}_0\right) \circ P_t(x),\, P_t(y)\right) + \left(\frac{1}{t} \cdot (P_t(x) - x),\, A_0(P_t(y))\right)\\
&\qquad + (\dot{A}_0(P_t(x) - x),\, P_t(y)) + (\dot{A}_0(x),\, P_t(y) - y),
\end{aligned}$$

and when $t \to 0$, $\|A_0(P_t(y))\| \to 0$, $\|(1/t) \cdot (P_t(x) - x)\|$ is bounded, and so these imply the assertion. Note here we used the fact that the family $\{P_t\}$ is of class C^1. Hence there exist $0 < \delta_0 \leq \delta$ and $0 < \varepsilon_0 \leq \varepsilon$ and for $0 < |t| \leq \delta_0$ the signatures coincide and A_t is an isomorphism between $P_t(H) = P_t(\operatorname{Ker} A_0)$ which gives the existences of the $p + q$ eigenvalues of the operator A_t, $0 < e_1(t) \leq e_2(t) \leq \cdots \leq e_p(t) \leq \varepsilon_0$ and $-\varepsilon_0 \leq e_{-q}(t) \leq e_{-q+1}(t) \leq \cdots \leq e_{-1}(t) < 0$. □

Proof of Proposition 3.22. By the assumption there are a complex number $\mathrm{e}^{\sqrt{-1}\theta_0}$ (close enough to $\mathrm{e}^{\sqrt{-1}\pi}$, but $\neq \mathrm{e}^{\sqrt{-1}\pi}$) and $\varepsilon > 0$ such that for $|t - t^*| \leq \varepsilon$ the operators $\mathrm{e}^{\sqrt{-1}\theta_0} - \mathbf{c}(t)$ are invertible and

$$\sum_{|\theta| \leq \varepsilon} \dim \operatorname{Ker}(\mathbf{c}(t) - \mathrm{e}^{\sqrt{-1}(\pi+\theta)}) < \infty.$$

Put $\mathbf{c}(t+t^*) = \mathbf{c}(t^*)\,\mathrm{e}^{\sqrt{-1}R_t}$ and let A_t be a selfadjoint operator defined by the transformation

$$A_t = \sqrt{-1}(\mathrm{e}^{\sqrt{-1}\theta_0} - \mathbf{c}(t + t^*))^{-1}(\mathrm{e}^{\sqrt{-1}\theta_0} + \mathbf{c}(t + t^*)) - \sqrt{-1}\frac{\mathrm{e}^{\sqrt{-1}\theta_0} - 1}{\mathrm{e}^{\sqrt{-1}\theta_0} + 1},$$

then $\{A_t\}_{|t| \leq \varepsilon}$ is a C^1-class family of Fredholm operators.

We have an expression of the derivative $\dot{A}_0$ by using the resolvent equation:

$$\dot{A}_0 = ((\mathrm{e}^{\sqrt{-1}\theta_0} - \mathbf{c}(t^*))^{-1})^* \circ 2\dot{R}_0 \circ (\mathrm{e}^{\sqrt{-1}\theta_0} - \mathbf{c}(t^*))^{-1}. \tag{3.21}$$

This shows that the derivatives $\dot{A}_0$ and $2\dot{R}_0$ are conjugate, which gives us the coincidence of the signatures of the two bilinear forms defined on $\mathrm{Ker}(\mathbf{c}(t^*) + \mathrm{Id}) = \mathrm{Ker}\, A_0$: for $x, y \in \mathrm{Ker}(\mathbf{c}(t^*) + \mathrm{Id}) = \mathrm{Ker}\, A_0$

$$\begin{aligned}&(((\mathrm{e}^{\sqrt{-1}\theta_0} - \mathbf{c}(t^*))^{-1})^* \circ 2\dot{R}_0 \circ (\mathrm{e}^{\sqrt{-1}\theta_0} - \mathbf{c}(t^*))^{-1}(x), y)\\ &\quad = \frac{2}{|\mathrm{e}^{\sqrt{-1}\theta_0} + 1|^2}(\dot{R}_0(x), y) = (\dot{A}_0(x), y).\end{aligned}$$

Hence by applying the preceding Lemma 3.23 to the operator family $\{A_t\}$ and returning back to the original family $\{\mathbf{c}(t)\}$ we have the desired numbers of positive and negative eigenvalues of the family $\{\mathbf{c}(t)\}_{|t|\leq\delta}$ for sufficiently small δ, which gives

$$\mathbf{M}(\{\mathbf{c}(t)\}_{|t|\leq\delta}) = \mathrm{sign}(\dot{A}_0) = \mathrm{sign}(\dot{R}_0) = \mathrm{sign}\, \tilde{Q}_{\mathfrak{M}}. \qquad \square$$

Remark 3.24. For crossing $t^* = 0$ or 1, we only consider the one-side differentiation in the definition of the crossing form. In these cases we have

$$\mathbf{M}(\{\mathbf{c}(t)\}_{0\leq t\leq\delta}) = -q, \qquad \mathbf{M}(\{\mathbf{c}(t)\}_{1-\delta\leq t\leq 1}) = p',$$

where the signature of $\tilde{Q}_{\mathfrak{M}}$ at $t^* = 0$ is (p, q) and at $t^* = 1$ (p', q').

Corollary 3.25. *Let $\mu : I \to \mathcal{F}\Lambda_\lambda(H)$ be a C^1-class path (so that $\mathcal{S}_\lambda \circ \mu(t)$ is a path in $\mathcal{U}_{\mathcal{F}}(H_J)$ also of class C^1). Let $0 < t^* < 1$ be a regular crossing of the curve $\{\mathcal{S}_\lambda \circ \mu(t)\}$. Then there exist a $\delta > 0$ such that*

$$\mathbf{Mas}(\{\mu(t)\}_{|t-t^*|\leq\delta}, \lambda) = \mathrm{sign}\, \tilde{Q}_{\mathfrak{M}},$$

where $\tilde{Q}_{\mathfrak{M}}$ denotes the crossing form of $\{\mathcal{S}_\lambda \circ \mu(t)\}$ at the time $t = t^*$.

There is another bilinear form (see [11,29]) for describing the Maslov index which will turn out to be more suitable for proving the *spectral flow formula* (see Section 6). It is based on a representation of μ as the graph of a suitable bounded operator. Let $\mu : I \to \mathcal{F}\Lambda_\lambda(H)$ be a path in $\mathcal{F}\Lambda_\lambda(H)$ of class C^1 and let $0 < t^* < 1$ be a crossing of the curve $\{\mathcal{S}_\lambda \circ \mu(t)\}$, i.e., $\mu(t^*) \cap \lambda \neq \{0\}$. For t, $|t - t^*| \ll 1$, $\mu(t)$ is transversal to $\mu(t^*)^\perp$ and in this neighborhood of t^*, each $\mu(t)$ can be written as the graph of a bounded operator $A_t : \mu(t^*) \to \mu(t^*)$, $\mu(t) = \{x + J \circ A_t(x) | x \in \mu(t^*)\}$. Note that the curve $\{A_t\}$ is also of class C^1. We consider the bilinear form

$$Q_{\mathfrak{M}}(x, y) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\omega(x, J \circ A_t(y))\right|_{t=t^*} \quad \text{for } x, y \in \mu(t^*). \tag{3.22}$$

In the above definition of the bilinear form $Q_{\mathfrak{M}}$ we used the fact that the inner product in the Hilbert space is compatible with the symplectic form ω, so that $\mu(t^*)^\perp$ is a Lagrangian subspace, But this is not essential. In fact, let ν be a Lagrangian subspace which is transversal to $\mu(t^*)$, then for sufficiently small $|t-t^*| \ll 1$ Lagrangian subspaces $\{\mu_t\}$ are transversal to ν. Then there is again a differentiable family of bounded operators $\{\phi_t\}_{|t-t^*|\ll 1}$, $\phi_t : \mu(t^*) \to \nu$, by which we have $\mu(t) =$ graph of ϕ_t for each t, $|t-t^*| \ll 1$.

Now let $y \in \mu(t^*)$, then we have

$$y + \phi_t(y) = z_t + J \circ A_t(z_t), \tag{3.23}$$

where $z_t = \mathcal{P}_{\mu(t^*)}(y + \phi_t(y))$ is a differentiable family in $\mu(t^*)$ and $z_{t^*} = y$. Hence by differentiating the both sides of (3.23) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_{t_{|t=t^*}}(y) = \mathcal{P}_{\mu(t^*)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\phi_{t_{|t=t^*}}(y)\right) + J \circ \frac{\mathrm{d}}{\mathrm{d}t}A_{t_{|t=t^*}}(y).$$

By this equality we have the invariance of the definition of the bilinear form $Q_{\mathfrak{M}}$ from the auxiliary fixed Lagrangian subspace ν.

Proposition 3.26.

$$Q_{\mathfrak{M}}(x, y) = \frac{\mathrm{d}}{\mathrm{d}t}\omega(x, \phi_t(y))_{|t=t^*}, \quad x, y \in \mu(t^*).$$

Proof. Let $x, y \in \mu(t^*)$, then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\omega(x, \phi_t(y))_{|t=t^*} &= \omega\left(x, \mathcal{P}_{\mu(t^*)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\phi_{t_{|t=t^*}}(y)\right) + J \circ \frac{\mathrm{d}}{\mathrm{d}t}A_{t_{|t=t^*}}(y)\right)\\ &= \omega\left(x, J \circ \frac{\mathrm{d}}{\mathrm{d}t}A_{t_{|t=t^*}}(y)\right) = Q_{\mathfrak{M}}(x, y).\end{aligned}$$ □

The bilinear form $Q_{\mathfrak{M}}$ is symmetric on $\mu(t^*)$ at each point t^* solely defined by the differentiable family $\{\mu_t\}$ itself and we show the coincidence of the signatures of two bilinear forms $Q_{\mathfrak{M}}$ and $\tilde{Q}_{\mathfrak{M}}$.

Proposition 3.27. *On* $\mu(t^*) \cap \lambda$, sign $Q_{\mathfrak{M}} =$ sign $\tilde{Q}_{\mathfrak{M}}$.

Proof. We have two expression of the space $\mu(t)$:

(a) $\mu(t) = \{x + J \circ A_t(x) | x \in \mu(t^*)\}$, $A_t \in \hat{\mathcal{B}}(\mu(t^*))$, $\{A_t\}$ is C^1-class.
(b) $\mu(t) = U_t(\lambda^{\perp})$, where $\{U_t\}$ is a C^1-class family of unitary operators on H_J.

Put $U_{t+t^*} = U_{t^*}\,\mathrm{e}^{\sqrt{-1}S_t}$ and $\mathbf{c}(t+t^*) = \mathcal{S}_\lambda(\mu(t+t^*)) = \mathbf{c}(t^*)\,\mathrm{e}^{\sqrt{-1}R_t}$, where $\{S_t\}$ and $\{R_t\}$ are C^1-class families of selfadjoint operators on H_J. We represent $S_t = X_t + \sqrt{-1}Y_t$ with $X_t, Y_t \in \mathcal{B}(\mu(t^*))$, $X = {}^tX$ and $Y = -{}^tY$.

By differentiating $\mathbf{c}(t) = U_t \circ \theta_\lambda(U_t)$ at $t = t^*$ we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{c}(t)_{|t=t^*} &= \dot{\mathbf{c}}(t^*) = U_{t^*} \circ \sqrt{-1}\dot{S}_0 \circ \theta_\lambda(U_{t^*}) + U_{t^*} \circ \theta_\lambda(U_{t^*} \circ \sqrt{-1}\dot{S}_0)\\ &= U_{t^*}(\sqrt{-1}(\dot{X}_0 + \sqrt{-1}\dot{Y}_0) + \sqrt{-1}(\dot{X}_0 - \sqrt{-1}\dot{Y}_0)) \circ \theta_\lambda(U_{t^*})\\ &= 2\sqrt{-1}U_{t^*} \circ \dot{X}_0 \circ \theta_\lambda(U_{t^*}) = \sqrt{-1}\mathbf{c}(t^*)\dot{R}_0.\end{aligned}$$

This identity says that the bilinear form $\tilde{Q}_{\mathfrak{M}}$ on H_J defined by $\dot{R}_0$ is unitary equivalent to the complexification of the bilinear form defined by the real selfadjoint operator $2\dot{X}_0$ on $\mu(t^*)$.

Now by differentiating the equality

$$U_t(x) = \mathcal{P}_{\mu(t^*)}(U_t(x)) + J \circ A_t(\mathcal{P}_{\mu(t^*)}(U_t(x))), \quad x \in \lambda^{\perp},$$

we have

$$\mathcal{P}_{\mu(t^*)} \circ \dot{S}_0(U_{t^*}(x)) = \dot{A}_0(U_{t^*}(x)).$$

Note that we used here the equation $J \circ \mathcal{P}_{\mu(t^*)} + \mathcal{P}_{\mu(t^*)} \circ J = J$.

Let $x, y \in \mu(t^*)$, then we have

$$\omega(x, J \circ \dot{A}_0(y)) = \langle x, \dot{A}_0(y)\rangle = \langle x, \mathcal{P}_{\mu(t^*)} \circ \dot{S}_0(y)\rangle = \langle x, \dot{X}_0(y)\rangle. \tag{3.24}$$

Hence the unitary equivalence (on the whole space H_J) of the bilinear forms defined by the operators $\dot{R}_0$ and $2\dot{X}_0$ and the Eq. (3.24) (note that the identity holds on $\mu(t^*)$) show the proposition. □

Remark 3.28. The unitary equivalence of the two bilinear forms $Q_{\mathfrak{M}}$ and $\tilde{Q}_{\mathfrak{M}}$ on $\mu(t^*)$ implies that the definition of the bilinear form $\tilde{Q}_{\mathfrak{M}}$ does not depend on the almost complex structure J by which we regard the real Hilbert space H as a complex Hilbert space H_J. This means we can freely replace the inner product by a suitable one. For example, we can assume that any two transversal Lagrangian subspaces are orthogonal (see proof of Theorem 5.10).

Now we have a similar formula with Proposition 3.22.

Corollary 3.29. *Let $\mu : I \to \mathcal{F}\Lambda_\lambda(H)$ be a C^1-class path. Let $0 < t^* < 1$ be a regular crossing of the curve. Then it is also regular crossing of the curve $\{\mathcal{S}_\lambda \circ \mu(t)\}$, and there exists a positive $\delta > 0$ such that*

$$\mathbf{Mas}(\{\mu(t)\}_{|t-t^*|\leq\delta}, \lambda) = \operatorname{sign} Q_{\mathfrak{M}},$$

where $Q_{\mathfrak{M}}$ denotes the crossing form of $\{\mu(t)\}$ at the time $t = t^*$.

Remark 3.30. In the paper [29] the authors gave a definition of the Maslov index (for the case of finite dimension) for such differentiable curves $\{\mathbf{c}(t)\}_{t\in[0,1]}$ that all their “crossings” are regular in terms of this bilinear form with corrections at the end points by adding the halves of the dimensions $\dim \lambda \cap \mathbf{c}(1)$ and $\dim \lambda \cap \mathbf{c}(0)$.

Finally in this subsection we give an example of a C^1-class path with a regular crossing and calculate the Maslov index.

Example 3.31. Let F be a finite dimensional subspace in $J(\lambda)$, and we define a family of unitary operators such that

$$U(t)(x) = \begin{cases} e^{\sqrt{-1}\pi t} \cdot x & x \in F \\ x & x \in F^{\perp} \cap J(\lambda) \end{cases} \tag{3.25}$$

For each $t, t \in [0, 1]$ let $\mu(t) = U(t)(J(\lambda))$, then $\mu(t) \in \mathcal{F}\Lambda_\lambda(H)$ and $t = 1/2$ is an only non-trivial crossing with λ and is regular. As is easily determined the crossing form is given by

$$Q_{\mathfrak{M}}(x, y) = \pi\langle x, y\rangle, \quad x, y \in J(F). \tag{3.26}$$

Hence we have

$$\mathbf{Mas}(\{\mu(t)\}_{0\le t\le 1}, \lambda) = \operatorname{sign} Q_{\mathfrak{M}} = \dim F.$$

Also for $0 < \epsilon \ll 1$

$$\mathbf{Mas}(\{\mu(t)\}_{1/2\le t\le 1/2+\epsilon}, \lambda) = 0,$$

and

$$\mathbf{Mas}(\{\mu(t)\}_{1/2-\epsilon\le t\le 1/2}, \lambda) = \dim F.$$

3.5. *Maslov index for paths of Fredholm pairs of Lagrangian subspaces*

In this subsection we will denote the direct sum of the symplectic Hilbert space (H, ω) and $(H, -\omega)$ with the notation $\mathbb{H} = H\boxplus H \equiv H_\omega\boxplus H_{-\omega}$. $\mathbb{H}$ is a symplectic Hilbert space with the symplectic form $\Omega = \omega - \omega$, and the corresponding almost complex structure $\mathbb{J} = J\boxplus - J$, so that we have $\mathbb{H}_{\mathbb{J}} = H_J\boxplus H_{-J}$. Then the diagonal Δ in $\mathbb{H}$ is a Lagrangian subspace. Let $\{(\mu_t, \lambda_t)\}_{t\in I}$ be a continuous family of Fredholm pairs of Lagrangian subspaces, then $\{\mu_t\boxplus\lambda_t\}$ is a curve in $\mathcal{F}\Lambda_\Delta(H_\omega\boxplus H_{-\omega})$. Of course it is natural to define the Maslov index of the curve of Fredholm pairs $\{(\mu_t, \lambda_t)\}$ to be $\mathbf{Mas}(\{\mu_t \oplus \lambda_t\}, \Delta)$.

Proposition 3.32. *Let $\{\mu_t\}$ be a continuous curve in $\mathcal{F}\Lambda_\lambda(H_\omega)$, then*

$$\mathbf{Mas}(\{\mu_t\}, \lambda) = \mathbf{Mas}(\{\mu_t \oplus \lambda\}, \Delta).$$

Remark 3.33. For loops this property will be well-known. For arbitrary continuous paths in the finite dimensional case this can be proved by making use of Proposition 4.3 (Section 4.1), but in the infinite dimensional case we have no such relations and we need a proof which is valid not only for loops but also for any continuous paths.

If we identify $\mathbb{H}_{\mathbb{J}} = \Delta + \Delta^\perp = \Delta + \mathbb{J}(\Delta) \cong \Delta \otimes \mathbb{C}$, then $\tau_\Delta(a\boxplus b) = b\boxplus a$. Let us decompose H as $H = \lambda \oplus \lambda^\perp$ and let $\varphi : \Delta \to \Delta^\perp$ be

$$\varphi((x, y)\boxplus(x, y)) = (-x, y)\boxplus(x, -y),$$

where we express elements in Δ by $(x, y)\boxplus(x, y)$, $x + y \in \lambda + \lambda^\perp = H$. Then we have

$$\operatorname{graph} \varphi = \lambda^\perp\boxplus\lambda.$$

Let $A = \mathbb{J} \circ \varphi : \Delta \to \Delta$ and $V : \mathbb{H}_{\mathbb{J}} \to \mathbb{H}_{\mathbb{J}}$ by

$$V = \frac{-\sqrt{-1}}{\sqrt{2}} - \frac{A \otimes \mathrm{Id}}{\sqrt{2}},$$

where we regard $A = A \otimes \mathrm{Id}$ is complexified according to the identification $\mathbb{H}_J \cong \Delta \otimes \mathbb{C}$. Then we have

$$\sqrt{-1}(A \otimes \mathrm{Id})((a, b)\boxplus(c, d)) = (c, -d)\boxplus(-a, b) \tag{3.27}$$

for $(a, b)\boxplus(c, d) \in H_J\boxplus H_{-J} = (\lambda + \lambda^{\perp})\boxplus(\lambda + \lambda^{\perp})$ and

$$V(\Delta^{\perp}) = \lambda^{\perp}\boxplus\lambda. \tag{3.28}$$

Now we define maps $\mathbf{a}_\lambda$, $\mathbf{b}_\lambda$ and P_λ as follows:

$$\mathbf{a}_\lambda : \mathcal{U}_\lambda(H_J) \to \mathcal{U}_\Delta(\mathbb{H}_{\mathbb{J}}), \qquad U \mapsto \tilde{U} \circ V,$$

where $\tilde{U} = U\boxplus\mathrm{Id} : H_J\boxplus H_{-J} \to H_J\boxplus H_{-J}$

$$\mathbf{b}_\lambda : \mathcal{U}_{\mathcal{F}}(H_J) \to \mathcal{U}_{\mathcal{F}}(\mathbb{H}_{\mathbb{J}}), \qquad W \mapsto \sqrt{-1} \cdot W \circ (A \otimes \mathrm{Id}),$$

and

$$P_\lambda : \mathcal{F}\Lambda_\lambda(H) \to \mathcal{F}\Lambda_\Delta(\mathbb{H}), \qquad \mu \mapsto \mu\boxplus\lambda.$$

Lemma 3.34. *The following diagram is commutative.*

$$\begin{tikzcd}
\mathcal{U}_\lambda(H_J) \arrow[r, "\mathbf{a}_\lambda"] \arrow[d, "\rho_\lambda"'] & \mathcal{U}_\Delta(H_J \boxplus H_{-J}) \arrow[d, "\rho_\Delta"] \\
\mathcal{F}\Lambda_\lambda(H) \arrow[r, "P_\lambda"'] \arrow[d, "\mathcal{S}_\lambda"'] & \mathcal{F}\Lambda_\Delta(H_\omega \boxplus H_{-\omega}) \arrow[d, "\mathcal{S}_\Delta"] \\
\mathcal{U}_{\mathcal{F}}(H_J) \arrow[r, "\mathbf{b}_\lambda"] & \mathcal{U}_{\mathcal{F}}(H_J \boxplus H_{-J})
\end{tikzcd}$$

Proof. It will be enough to prove $\mathcal{S}_\Delta \circ P_\lambda = \mathbf{b}_\lambda \circ \mathcal{S}_\lambda$. Since $\theta_\Delta(V) = V$, $V^2 = \sqrt{-1} \cdot A \otimes \mathrm{Id}$ and $\theta_\Delta(\tilde{U}) = \mathrm{Id}\boxplus U^*$ we have

$$\begin{aligned}\mathcal{S}_\Delta \circ \rho_\Delta(\mathbf{a}_\lambda(U)) &= \tilde{U} \circ V \circ \tau_\Delta \circ (\tilde{U} \circ V)^* \circ \tau_\Delta = \tilde{U} \circ \sqrt{-1}(A \otimes \mathrm{Id}) \circ \theta_\Delta(\tilde{U}) \\ &= \widetilde{U \circ \theta_\lambda}(U) \circ \sqrt{-1}(A \otimes \mathrm{Id}),\end{aligned}$$

which prove the commutativity of the diagram. □

Proof of Proposition 3.32. From the above lemma we can show that if E is an eigenvalue of $\mathcal{S}_\Delta(\rho_\Delta(\mathbf{a}_\lambda(U)))$, then $-E^2$ is an eigenvalue of $\mathcal{S}_\lambda \circ \rho_\lambda(U)$. Conversely if $l = \mathrm{e}^{\sqrt{-1}\sigma}$ is an eigenvalue of $\mathcal{S}_\lambda \circ \rho_\lambda(U)$, then only one of $\pm\mathrm{e}^{\sqrt{-1}(\pi+\sigma)}$ is closed to -1. So if we have a continuous curve $\{\mu_t\} \subset \mathcal{F}\Lambda_\lambda(H)$, then the numbers of eigenvalues of $\{\mathcal{S}_\lambda(\mu_t)\}$ and $\{\mathcal{S}_\Delta(\mu_t\boxplus\lambda)\}$ which across $\mathrm{e}^{\sqrt{-1}\pi}$ coincide in both directions. This proves the proposition. □

The next property will be also natural.

Proposition 3.35. $\mathbf{Mas}(\{\mu_t\boxplus\lambda_t\}, \Delta) = -\mathbf{Mas}(\{\lambda_t\boxplus\mu_t\}, \Delta)$.

4. Finite dimensional cases

In the finite dimensional cases, the Maslov index for arbitrary continuous paths in the Lagrangian–Grassmannian was first defined in the paper [16] by noting the extendibility of the "*Leray index*" for arbitrary pairs of points on the universal covering space of the Lagrangian–Grassmannian by making use of the cocycle condition of the "*Leray index*", and this condition comes from the relation with the "*Kashiwara index*". Conversely, first we define Maslov index for arbitrary paths with respect to a Maslov cycle as we gave above, then we can define the "*Leray index*" for arbitrary pairs of points on the universal covering of the Lagrangian–Grassmannian (Proposition 4.3).

In the infinite dimensional case we could define the Maslov index for arbitrary paths with respect to a Maslov cycle as we did in the above Definition 3.8, however we cannot define "*Kashiwara index*" for arbitrary triples of Lagrangian subspaces like the finite dimensional case, although we have a symmetric bilinear form similar to the finite dimensional case. Only we can define it for mutually almost coincident triples, since then the symmetric bilinear form is of finite rank. Also we cannot define "Leray index" for arbitrary pairs of points on the universal covering space of the Fredholm–Lagrangian–Grassmannian.

In this section, following [16] we summarize the mechanism for defining the Maslov index for paths in $\Lambda(H)$ of the finite dimensional symplectic vector space H, and give an extension of the "Kashiwara index" to arbitrary triples of unitary operators (Section 4.2).

4.1. Leray index and Kashiwara index

Let ℓ_1, ℓ_2 and ℓ_3 be three Lagrangian subspaces and define the quadratic form Q on the direct sum $\ell_1 \oplus \ell_2 \oplus \ell_3$ as follows:

$$Q(x, x', x'') = \omega(x, x') + \omega(x', x'') + \omega(x'', x), \quad x \in \ell_1, \ x' \in \ell_2, \ x'' \in \ell_3. \tag{4.1}$$

The corresponding bilinear form is

$$\begin{aligned} &I_\omega(x, x', x''; a, a', a'') \\ &\quad = \omega(x, a') + \omega(x', a) + \omega(x, a'') + \omega(x'', a) + \omega(x', a'') + \omega(x'', a'), \\ &x, a \in \ell_1, \ x', a' \in \ell_2, \ x'', a'' \in \ell_3. \end{aligned}$$

The signature of this quadratic form is called "*Kashiwaka index*" or "*cross index*" of the triple of Lagrangian subspaces. We denote it by $\sigma(\ell_1, \ell_2, \ell_3)$.

In the finite dimension cases, although $\Lambda_\lambda(H) = \Lambda(H)$ always, it should be noted that the Souriau map $\mathcal{S}_\lambda : \Lambda(H) \to \mathcal{U}(H_J)$ itself depends on the pre-fixed Lagrangian subspace λ. Now let

$$\tilde{\mathcal{U}}(H_J) = \{(U, \alpha) \in \mathcal{U}(H_J) \times \mathbb{R} | \det U = \mathrm{e}^{\sqrt{-1}\alpha}\}$$

be a realization of the universal covering of the unitary group $\mathcal{U}(H_J)$, then the space $\mathcal{S}_\lambda^*(\tilde{\mathcal{U}}(H_J)) = \tilde{\Lambda}(H) = \{(\mu, \alpha) \in \Lambda(H_J) \times \mathbb{R} | \det \mathcal{S}_\lambda(\mu) = \mathrm{e}^{\sqrt{-1}\alpha}\}$ is the universal covering of the Lagrangian–Grassmannian $\Lambda(H)$ with the projection map $q_\lambda : \tilde{\Lambda}(H) \to \Lambda(H)$. Let $\tilde{\ell}_1$ and $\tilde{\ell}_2$ be two point on $\mathcal{S}_\lambda^*(\tilde{\mathcal{U}}(H_J)) = \tilde{\Lambda}(H)$ and we assume that $q_\lambda(\tilde{\ell}_1) = \ell_1$ and

$q_\lambda(\tilde{\ell}_2) = \ell_2$ are transversal, i.e., $\ell_1 \cap \ell_2 = \{0\}$. Then $\mathrm{Id} + \mathcal{S}_\lambda(\ell_1)\mathcal{S}_\lambda(\ell_2)^*$ is invertible, so we define "*Leray index*" $\mu(\tilde{\ell}_1, \tilde{\ell}_2)$ of such a pair by the following definition.

Definition 4.1.

$$\mu(\tilde{\ell}_1, \tilde{\ell}_2) = \frac{1}{2\pi}(\alpha_1 - \alpha_2 + \sqrt{-1}\,\mathrm{Tr}\log(-\mathcal{S}_\lambda(\ell_1)\circ\mathcal{S}_\lambda(\ell_2)^*)),$$

where log is defined by (3.18).

We have a fundamental relation of the Leray index and the Kashiwara index (cocycle condition of the Leray index).

Proposition 4.2. *Let $\tilde{\ell}_1$, $\tilde{\ell}_2$ and $\widetilde{\ell_3}$ be three points on $\tilde{\Lambda}(H_J)$ such that each of the pairs $(q_\lambda(\ell_1), q_\lambda(\ell_2))$, $(q_\lambda(\ell_1), q_\lambda(\ell_3))$ and $(q_\lambda(\ell_2), q_\lambda(\ell_3))$ is transversal, then*

$$\mu(\tilde{\ell}_1, \tilde{\ell}_2) + \mu(\tilde{\ell}_2, \widetilde{\ell_3}) + \mu(\widetilde{\ell_3}, \tilde{\ell}_1) = \sigma(q_\lambda(\ell_1), q_\lambda(\ell_2), q_\lambda(\ell_3)). \tag{4.2}$$

Then we define for any pairs $(\tilde{\ell}_1, \tilde{\ell}_2) \in \tilde{\Lambda}(H) \times \tilde{\Lambda}(H)$ (without transversality assumption between the pair $(q_\lambda(\tilde{\ell}_1), q_\lambda(\tilde{\ell}_2))$) the "*Leray index*" $\mu(\tilde{\ell}_1, \tilde{\ell}_2)$ by the formula

$$\mu(\tilde{\ell}_1, \tilde{\ell}_2) = \mu(\tilde{\ell}, \tilde{\ell}_2) - \mu(\tilde{\ell}, \tilde{\ell}_1) + \sigma(q_\lambda(\ell_1), q_\lambda(\ell_2), q_\lambda(\ell)) \tag{4.3}$$

by taking an element $\tilde{\ell}$ in $\tilde{\Lambda}(H)$ such that $q_\lambda(\tilde{\ell})$ is transversal to each $q_\lambda(\widetilde{\ell_i})$ $(i = 1, 2)$.

The independence of this value from the choice of such $\tilde{\ell}$ is proved by making use of the fact

$$\partial\sigma(\ell_0, \ell_1, \ell_2, \ell_3) = \sigma(\ell_1, \ell_2, \ell_3) - \sigma(\ell_0, \ell_2, \ell_3) + \sigma(\ell_0, \ell_1, \ell_3) - \sigma(\ell_0, \ell_1, \ell_2) = 0. \tag{4.4}$$

Now we fix a $\lambda \in \Lambda(H)$ and let $\{\mathbf{c}(t)\}_{t\in[0,1]}$ be a continuous curve in $\Lambda(H)$, and take a lifting $\{\tilde{\mathbf{c}}(t)\}_{t\in[0,1]}$ of the curve $\{\mathbf{c}(t)\}$. Then we have the following proposition.

Propositiion 4.3.

$$\mathbf{Mas}(\{\mathbf{c(t)}\}, \lambda) = \mu(\tilde{\mathbf{c}}(0), \tilde{\mathbf{c}}(1)) - \sigma(\lambda, \mathbf{c}(1), \mathbf{c}(0)).$$

4.2. Complex Kashiwara index

By the very definition of the Leray index we see that it can be defined, by the same formula, for any pairs of the points $((U, \alpha_1), (U, \alpha_2))$ in $\tilde{U}(H_J)$ with the property that $\mathrm{Id} + U_1 \circ U_2^{-1}$ is invertible:

$$\mu((U, \alpha_1), (U, \alpha_2)) = \frac{1}{2\pi}(\alpha_1 - \alpha_2 + \sqrt{-1}\,\mathrm{Tr}\log(-U_1 \circ U_2^{-1})).$$

Then for such triples the sum

$$\mu((U, \alpha_1), (U, \alpha_2)) + \mu((U, \alpha_2), (U, \alpha_3)) + \mu((U, \alpha_3), (U, \alpha_1))$$

is independent of $(\alpha_1, \alpha_2, \alpha_3)$. This enables us to extend the Kashiwara index for any triples of unitary matrices. We explain it here.

We denote the sesquilinear extension of the symplectic form ω by $\omega^{\mathbb{C}}$.

For three Lagrangian subspaces $L_i \in \Lambda^{\mathbb{C}}(H \otimes \mathbb{C})$ $(i = 1, 2, 3)$ we define the bilinear form

$$I^{\mathbb{C}}_{\omega} : L_1 \oplus L_2 \oplus L_3 \times L_1 \oplus L_2 \oplus L_3 \to \mathbb{C}, \tag{4.5}$$

by

$$\begin{aligned} &I^{\mathbb{C}}_{\omega}(z, z', z''; w, w', w'') \\ &\quad = \omega^{\mathbb{C}}(z, w') + \omega^{\mathbb{C}}(w, z') + \omega^{\mathbb{C}}(z, w'') + \omega^{\mathbb{C}}(z'', w) + \omega^{\mathbb{C}}(z', w'') + \omega^{\mathbb{C}}(z'', w'), \\ &z = x + \sqrt{-1}y, \quad w = a + \sqrt{-1}b \in L_1, \quad z', w' \in L_2, \quad z'', w'' \in L_3. \end{aligned}$$

Then this is an Hermite form on $L_1 \oplus L_2 \oplus L_3$. We denote the signature of this form by

$$\sigma^{\mathbb{C}}(L_1, L_2, L_3)$$

and call it "complex Kashiwara index" or "complex cross index". If each $L_i = \lambda_i \otimes \mathbb{C}$ with $\lambda_i \in \Lambda(H)$, then this is the sesquilinear extension of the bilinear form I_ω and their signatures coincide.

Next recall the isomorphism Φ_λ (2.46):

$$\Phi_\lambda : \mathcal{U}(H_J) \to \Lambda^{\mathbb{C}}(H \otimes \mathbb{C}),$$

$$\Phi_\lambda(V) = \text{the graph of the unitary operator} - \mathfrak{k} \circ V \circ \tau_\lambda \circ \mathfrak{K}^{-1} \in \mathcal{U}(E_+, E_-).$$

Proposition 4.4. *Let* $(U_i, \alpha_i) \in \tilde{\mathcal{U}}(H_J)$, $i = 1, 2, 3$, *i.e.*, $\det U_i = \mathrm{e}^{\sqrt{-1}\alpha_i}$, *and assume* $\det(U_i \circ U_j^{-1} + \mathrm{Id}) \neq 0$, $i, j = 1, 2, 3$, then

$$\begin{aligned} &\mu((U_1, \alpha_1), (U_2, \alpha_2)) + \mu((U_2, \alpha_2), (U_3, \alpha_3)) + \mu((U_3, \alpha_3), (U_1, \alpha_1)) \\ &\quad = \sigma^{\mathbb{C}}(\Phi_\lambda(U_1), \Phi_\lambda(U_2), \Phi_\lambda(U_3)). \end{aligned}$$

Especially the value of $\sigma^{\mathbb{C}}$ *does not depend on the fixed* λ.

Now let (U_1, α_1) and (U_2, α_2) be any pair of the points in $\tilde{\mathcal{U}}(H_J)$ and choose an element (U, α) such that

$$\det(\mathrm{Id} + U_1 \circ U^{-1}) \neq 0 \quad \text{and} \quad \det(\mathrm{Id} + U_2 \circ U^{-1}) \neq 0. \tag{4.6}$$

Then we can define the "Leray index" of the pair (U_1, α_1), (U_2, α_2) by the formula.

Definition 4.5.

$$\begin{aligned} &\mu((U_1, \alpha_1), (U_2, \alpha_2)) \\ &\quad = \mu((U, \alpha), (U_2, \alpha_2)) - \mu((U, \alpha), (U_1, \alpha_1)) + \sigma^{\mathbb{C}}(\Phi_\lambda(U_1), \Phi_\lambda(U_2), \Phi_\lambda(U)). \end{aligned}$$

The similar cocycle property of $\sigma^{\mathbb{C}}$ with (4.4) guarantees that this definition does not depend on the choice of the element (U, α) which satisfies the condition (4.6), and the function is defined on the space $\tilde{\mathcal{U}}(H) \times \tilde{\mathcal{U}}(H)$. Also it is the extension of the Leray index defined on the space $\tilde{\Lambda}(H) \times \tilde{\Lambda}(H)$ through the embedding

$$\tilde{\Lambda}(H) \times \tilde{\Lambda}(H) \hookrightarrow \tilde{\mathcal{U}}(H_J) \times \tilde{\mathcal{U}}(H_J).$$

Note that this embedding is defined by choosing a $\lambda \in \Lambda(H)$.

Remark 4.6. We do not state the invariance of the Maslov index and other indexes (finite and infinite dimensions) under the unitary and symplectic group actions. These will be proved by making use of the properties of the Souriau map.

5. Polarization and a reduction theorem of the Maslov index

For the proof of Theorem 2.54 we employed the finite dimensional reduction (= the diagram (2.44)) of the Maslov index from infinite dimensions to finite dimensions. In this section we prove a reduction Theorem 5.10 of the Maslov index inside the infinite dimensions.

5.1. Symplectic transformation and Canonical relation

First we remark a continuity of a symplectic transformation.

Let H_i $(i = 0, 1)$ be two symplectic Hilbert spaces equipped with compatible symplectic structures (symplectic forms ω_i, inner products $\langle \bullet, \bullet \rangle_i$ and almost complex structures J_i $(i = 0, 1)$). As in Section 3.5 we consider the direct sum $H_0 \boxplus H_1$ as a symplectic Hilbert space with the compatible symplectic form

$$\Omega((x, a), (y, b)) = \omega_0(x, y) - \omega_1(a, b), \quad (x, a), (y, b) \in H_0 \boxplus H_1.$$

Let $S : H_0 \to H_1$ be a linear map defined on the whole space H_0 and we assume that S keeps the symplectic forms:

$$\omega_1(S(x), S(y)) = \omega_0(x, y), \quad \text{for all } x \text{ and } y \in H_0. \tag{5.1}$$

Then it is easy to see that S is injective and the graph $G_S = \{(x, S(x)) | x \in H_0\}$ is an isotropic subspace. Under the assumption (5.1), the closure of the image $S(H_0)$ is a symplectic Hilbert space. So now we start assuming that S has a dense image, then we have the following proposition.

Proposition 5.1. *The graph G_S is a Lagrangian subspace in $H_0 \boxplus H_1$. Hence S is a bounded operator by the closed graph theorem.*

Proof. Let $(a, b) \in H_0 \boxplus H_1$ and $\Omega((a, b), (x, S(x))) = 0$ for all $x \in H_0$. Then we have $\omega_0(x, a) = \omega_1(S(x), b) = \omega_1(S(x), S(a))$. Hence $S(a) = b$, which shows that the graph G_S is a Lagrangian subspace in $H_0 \boxplus H_1$. □

By this proposition, if a symplectic transformation S between two symplectic Hilbert spaces is algebraically isomorphic, then it must be topologically isomorphic.

We call a Lagrangian subspace C in the direct sum $H_0 \boxplus H_1$ with the symplectic form $\Omega = \omega_0 - \omega_1$ a canonical relation (see for the global settings [20,21]). The graph of a symplectic transformation defined on the whole space H_0 with the dense image is so a canonical relation.

In this section we consider a particular canonical relation given as a graph of a closed symplectic transformation, that is, let S be a densely defined closed and symplectic transformation, and in particular not continuous:

$$S : \mathfrak{D}_S \to H_1, \omega_0(x, y) = \omega_1(S(x), S(y)), \quad x \text{ and } y \in \mathfrak{D}_S,$$

where $\mathfrak{D}_S$ is a dense subspace in the symplectic Hilbert space H_0. Then again we have the following proposition.

Proposition 5.2. *Let us assume that S has the dense image, then the graph G_S is a Lagrangian subspace in $H_0 \boxplus H_1$.*

Let $\lambda \in \Lambda(H_0)$, then we have always $\lambda \cap \mathfrak{D}_S \neq \{0\}$ and $S(\lambda)$ is an isotropic subspace, but will not be a Lagrangian subspace in general.

In the next subsection we will show Theorem 5.10 that if we induce this map S between certain Fredholm–Lagrangian–Grassmannians, then it preserves the Maslov index under additional assumptions.

5.2. Polarization of symplectic Hilbert spaces

Again let H be a symplectic Hilbert space.

Definition 5.3. We say that the symplectic Hilbert space is polarized, when H is decomposed into a direct sum of two Lagrangian subspaces

$$H = \ell_+ \oplus \ell_-.$$

Or we say that the sum $H = \ell_+ \oplus \ell_-$ is a polarization of H.

Remark 5.4. In the polarization $H = \ell_+ \oplus \ell_-$ the subspaces need not be orthogonal, however it is possible by replacing the inner product (*symplectic form should not be changed always*) that the sum is orthogonal. Of course the new norm is equivalent to the initial one (Remark 3.28)).

Proposition 5.5. *Let $H = \ell_- + \ell_+$ be a polarized symplectic Hilbert space.*

(a) *Let S be a closed subspace in ℓ_-, and we take a complement T of S in ℓ_-, $\ell_- = S + T$. Put $F = T^\circ \cap \ell_+$, then $S + F$ is a symplectic subspace, in fact*

$$(S + F)^\circ = T + S^\circ \cap \ell_+$$

and

$$S + F + T + S^\circ \cap \ell_+ = H.$$

(b) *Let S be a closed subspace in ℓ_- and F be a closed subspace in ℓ_+. Assume that $S+F$ is a symplectic subspace, then $F^\circ \cap \ell_- + S^\circ \cap \ell_+$ is a symplectic subspace*

$$(S+F)^\circ = F^\circ \cap \ell_- + S^\circ \cap \ell_+.$$

Proof. Proof of (a). Put $G = S^\circ \cap \ell_+$, then

$$(S+F)^\circ = S^\circ \cap (T^\circ \cap \ell_+)^\circ = T + S^\circ \cap \ell_+ = T + G,$$

since $T \subset S^\circ$. Next by $F + G = T^\circ \cap \ell_+ + S^\circ \cap \ell_+ \subset \ell_+$ we have $(T^\circ \cap \ell_+ + S^\circ \cap \ell_+)^\circ = (T + \ell_+) \cap (S + \ell_+) = \ell_+$, hence $F + G = \ell_+$. So $F + S + G + T = H$.

Proof of (b). Put $T = F^\circ \cap \ell_-$ and $G = S^\circ \cap \ell_+$, then $(T + G)^\circ = (F + \ell_-) \cap (S + \ell_+) = F + S$. Now it is enough to show that $G + F = \ell_+$. Since $(G + F) \subset \ell_+$, $\ell_+ \subset (F + S^\circ \cap \ell_+)^\circ = F^\circ \cap (S + \ell_+) = F^\circ \cap S + \ell_+ = \ell_+$. Here we used the fact that $\ell_+ \subset F^\circ$ and $F^\circ \cap S = \{0\}$. This proves (b). □

Corollary 5.6. *Let $\lambda_- + \lambda_+ = H$ be a polarization and $S + T = \lambda_-$ be a decomposition by closed subspaces. Put $F = T^\circ \cap \lambda_+$ and $G = S^\circ \cap \lambda_+$, then we have a new polarization of H, $H = (T + F) + (S + G)$.*

Corollary 5.7. *Let $H = \ell_- + \ell_+$ be a polarized symplectic Hilbert space, S a closed subspace in ℓ_- and F a closed subspace in ℓ_+. Assume that $S + F$ is a symplectic subspace as in the proposition above* (b)*, then we can introduce a compatible inner product with the symplectic form which satisfies that all for isotropic subspaces S, F, $T = F^\circ \cap \ell_-$ and $G = S^\circ \cap \ell_+$ are mutually orthogonal.*

We use this corollary in the proof of Theorem 5.10.

Remark 5.8. The operation $S \mapsto S^\circ \cap \ell_+$ is idempotent. In fact, $(S^\circ \cap \ell_+)^\circ \cap \ell_- = (S + \ell_+) \cap \ell_- = S$.

Let us consider the following situation:

[**CP1**]: *Let H and B be symplectic Hilbert spaces with a compatible symplectic structure ω_H and ω_B, respectively. We assume both are polarized with Lagrangian subspaces $\lambda_\pm$ and $\ell_\pm$:*

$$B = \lambda_+ + \lambda_-, \qquad H = \ell_+ + \ell_-.$$

[**CP2**]: *There are continuous injective maps*

$$\mathbf{i}_+ : \ell_+ \to \lambda_+, \qquad \mathbf{i}_- : \lambda_- \to \ell_-$$

having "dense images".

[**CP3**]: *For any $x \in \ell_+$ and $b \in \lambda_-$*

$$\omega_B(\mathbf{i}_+(x), b) = \omega_H(x, \mathbf{i}_-(b)).$$

Remark 5.9. An example satisfying these conditions is given in Section 6.1, Example 6.7.

Let $\mu \in \mathcal{F}\Lambda(B)$ and put

$$\boldsymbol{\gamma}(\mu) = \{(x, y) \in \ell_+ \oplus \ell_- | \exists b \in \lambda_-, (\mathbf{i}_+(x), b) \in \mu \text{ and } y = \mathbf{i}_-(b)\}.$$

The subspace $\boldsymbol{\gamma}(\mu)$ is always isotropic, but need not be always Lagrangian nor closed. For example, if we take λ_- as μ, then $\boldsymbol{\gamma}(\lambda_-) = \mathbf{i}_-(\lambda_-)$ is dense but not closed in ℓ_-, so it is not a Lagrangian subspace. However if we restrict the map $\boldsymbol{\gamma}$ to $\mathcal{F}\Lambda_{\lambda_-}(B)$, then we have the following theorem.

Theorem 5.10.

(a) *For* $\mu \in \mathcal{F}\Lambda_{\lambda_-}(B)$, $\boldsymbol{\gamma}(\mu) \in \mathcal{F}\Lambda_{\ell_-}(H)$.

(b) *The map* $\boldsymbol{\gamma} : \mathcal{F}\Lambda_{\lambda_-}(B) \to \mathcal{F}\Lambda_{\ell_-}(H)$ *is continuous* (*and more strongly it is differentiable*).

(c) *Let* $\{\mathbf{c}(t)\}$ *be a continuous curve in* $\mathcal{F}\Lambda_{\lambda_-}(B)$, *then*

$$\mathbf{Mas}(\{\mathbf{c}(t)\}, \lambda_-) = \mathbf{Mas}(\{\boldsymbol{\gamma}(\mathbf{c}(t))\}, \ell_-). \tag{5.2}$$

We prove this theorem in the next subsection.

Let $\mathfrak{D} = \mathbf{i}_+(\ell_+) + \lambda_-$ and $\mathbf{S} : \mathfrak{D} \to H$, $\mathbf{S} : \mathbf{i}_+(x) + b \mapsto x + \mathbf{i}_-(b)$, then by the assumption [**CP3**] the map $\mathbf{S}$ is a symplectic transformation and we have the following proposition.

Proposition 5.11. *Let* $\mathbb{H} = H \boxplus B$ *be the symplectic Hilbert space with the symplectic form* $\Omega = \omega_H - \omega_B$ *and let*

$$C = \{(x, y) \boxplus (a, b) \in \mathbb{H} | x \in \ell_+, \quad b \in \lambda_-, \quad y = \mathbf{i}_-(b), \quad a = \mathbf{i}_+(x)\}.$$

Then C *is the graph of the map* $\mathbf{S}$, *is a Lagrangian subspace and* $\mathbf{S}(\mu) = \boldsymbol{\gamma}(\mu)$.

Proof. It is easy to show that C is isotropic. So we only prove the following: let $(u, v) \boxplus (k, k') \in \mathbb{H}$ satisfying

$$\Omega((u, u') \boxplus (k, k'), (x, \mathbf{i}_-(b)) \boxplus (\mathbf{i}_+(x), b)) = 0$$

for any $(x, \mathbf{i}_-(b)) \boxplus (\mathbf{i}_+(x), b) \in C$.

Then we have

$$\omega_H(u, \mathbf{i}_-(b)) + \omega_H(u', x) - \omega_B(k, b) - \omega_B(k', \mathbf{i}_+(x)) = 0.$$

From this equation and Assumption [**CP3**], we have $u' - \mathbf{i}_-(k') = 0$ and $k - \mathbf{i}_+(u) = 0$, which show $(u, u') \boxplus (k, k') \in C$ and C is a Lagrangian subspace.

Now we see the coincidence $\mathbf{S}(\mu) = \boldsymbol{\gamma}(\mu)$ by the definitions. □

5.3. Proof of Theorem 4.10

Let S be a finite dimensional subspace in λ_-, then there is a finite dimensional subspace L in ℓ_+ such that $\mathbf{i}_-(S) + L$ is a symplectic subspace in H and by Assumption [**CP3**] $S + \mathbf{i}_+(L)$ is also a symplectic subspace in B. There are many possibility to choose such a subspace L. We fix one of them by introducing a compatible inner product in H with respect to which $\ell_\pm$ are orthogonal. So we put $L = J_H(\mathbf{i}_-(S))$, where J_H is the almost complex structure defined by the compatible inner product.

Hereafter we put the symplectic subspaces $S + \mathbf{i}_+(L) = B_S$ and $\mathbf{i}_-(S) + L = H_S$ with $L = J_H(\mathbf{i}_-(S))$, corresponding to a finite dimensional subspace B in λ_-.

Then we have a symplectic isomorphism

$$\mathbf{i}^S = \mathbf{i}_{-|B_S} \oplus (\mathbf{i}_+)^{-1}_{|\mathbf{i}_+(L)} : B_S = S + \mathbf{i}_+(L) \xrightarrow{\sim} H_S = \mathbf{i}_-(S) + L,$$
$$x + y \mapsto \mathbf{i}_-(x) + (\mathbf{i}_+)^{-1}(y).$$

Next we remark that if θ is a Lagrangian subspace in B_S then $(S^\perp \cap \lambda_-) + \theta$ is a Lagrangian subspace in B. Let us denote the Lagrangian subspace of the form $(S^\perp \cap \lambda_-) + \theta$ by $\lambda(S, \theta)$ and denote the subset of such Lagrangian subspaces that are "transversal" to λ_+ by

$$\Lambda_S^{(0)} = \{\lambda(S, \theta) = (S^\perp \cap \lambda_-) + \theta | \lambda(S, \theta) \cap \lambda_+ = \{0\}, \theta \in \Lambda(B_S)\}.$$

Then we have

$$\bigcup_{B\subset\lambda_-, \dim B<\infty} \ \bigcup_{\lambda\in\Lambda_S^{(0)}} \mathcal{F}\Lambda_\lambda^{(0)}(B) = \mathcal{F}\Lambda_{\lambda_-}(B).$$

For $\mu \in \mathcal{F}\Lambda_{\lambda_-}(B)$ put $S = \lambda_- \cap \mu$, then there is a Lagrangian subspace θ in B_S such that the Lagrangian subspace of the form $(S^\perp \cap \lambda_-) + \theta$ is transversal both to μ and λ_+ by Proposition 2.43.

Let $\lambda(S, \theta) \in \Lambda_S^{(0)}$, then we can define new polarizations of H and B which satisfy Assumptions [**CP1**], [**CP2**] and [**CP3**] by making use of $\lambda(S, \theta)$. Theses are obtained by replacing the Lagrangian subspaces λ_- and ℓ_- by $\lambda(S, \theta)$ and $((\mathbf{i}_-(S))^\perp \cap \ell_-) + \mathbf{i}^S(\theta)$ respectively with the same λ_+ and ℓ_+. We replace the map $\mathbf{i}_-$ by $\mathbf{i}_{-|S^\perp\cap\lambda_-} \oplus \mathbf{i}^S_{|\theta}$. Note that this change of polarizations do not change the map $\boldsymbol{\gamma}$.

Now let $\lambda \in \Lambda_S^{(0)}$, then any μ in $\mathcal{F}\Lambda_\lambda^{(0)}(B)$ is expressed as a graph of a continuous map $\phi_\mu : \lambda_+ \to \lambda$. Hence on $\mathcal{F}\Lambda_\lambda^{(0)}(B)$ the map $\boldsymbol{\gamma}$ is expressed in the form

$$\boldsymbol{\gamma}(\mu) = \text{the graph of the map } \mathbf{i}^S \circ \phi_\mu \circ \mathbf{i}_+.$$

So it will be apparent of the continuity and also of the differentiability of the map $\boldsymbol{\gamma}$ on $\mathcal{F}\Lambda_\lambda^{(0)}(B)$, if we know $\boldsymbol{\gamma}(\mu) \in \mathcal{F}\Lambda_{\ell_-}(H)$.

To show the last assertion it is enough to prove the case when λ is λ_- and also it will be enough to prove $\gamma(\mu)^\circ \subset \gamma(\mu)$. So let $x \in \ell_+$ be an arbitrary element and assume that an element $a + b \in \ell_+ + \ell_-$ satisfies

$$\omega_H(x + \mathbf{i}_- \circ \phi \circ \mathbf{i}_+(x), a + b) = 0.$$

Then we have

$$\omega_H(x, b) + \omega_H(\mathbf{i}_- \circ \phi \circ \mathbf{i}_+(x), a) = 0.$$

Hence by Assumption [**CP3**]

$$\begin{aligned}
&\omega_H(x, b) + \omega_H(\mathbf{i}_- \circ \phi \circ \mathbf{i}_+(x), a)\\
&\quad = \omega_H(x, b) + \omega_B(\phi \circ \mathbf{i}_+(x), \mathbf{i}_+(a))\\
&\quad = \omega_H(x, b) + \omega_B(\phi \circ \mathbf{i}_+(x), \mathbf{i}_+(a)) + \omega_B(\mathbf{i}_+(x), \phi \circ \mathbf{i}_+(a)) - \omega_H(x, \mathbf{i}_- \circ \phi \circ \mathbf{i}_+(a))\\
&\quad = \omega_H(x, b) + \omega_B(\mathbf{i}_+(x) + \phi \circ \mathbf{i}_+(x), \mathbf{i}_+(a) + \phi \circ \mathbf{i}_+(a)) - \omega_H(x, \mathbf{i}_- \circ \phi \circ \mathbf{i}_+(a))\\
&\quad = \omega_H(x, b) - \omega_H(x, \mathbf{i}_- \circ \phi \circ \mathbf{i}_+(a)) = 0.
\end{aligned}$$

From this we have

$$b - \mathbf{i}_- \circ \phi \circ \mathbf{i}_+(a) = 0,$$

which shows that $(a, b) = (a, \mathbf{i}_- \circ \phi \circ \mathbf{i}_+(a)) \in \gamma(\mu)$. Hence $\boldsymbol{\gamma}(\mu)$ is a Lagrangian subspace. These prove Theorem 5.10(a) and (b).

For the proof of Theorem 5.10(c), we compute Maslov indexes of particular paths explicitly along the following steps (F1)–(F4) (see also similar arguments in [8]):

(F1) Let $\{\mathbf{c}(t)\}$ be a path such that all $\mathbf{c}(t)$ is transversal to λ_-, then $\boldsymbol{\gamma}(\mathbf{c}(t))$ is also transversal to ℓ_- for any t. Hence $\mathbf{Mas}(\{\mathbf{c}(t)\}, \lambda_-) = \mathbf{Mas}(\{\boldsymbol{\gamma}(\mathbf{c}(t))\}, \ell_-) = 0$.

(F2) Let L be a subspace in ℓ_+ with $\dim L = 1$ and let $\{\mathbf{c}(t)\}$ be a loop defined in Example 3.31 of Section 3.4 for $F = \mathbf{i}_+(L)$. We assume the symplectic structure in B is compatible. So $F + J_B(F)$ is symplectic and $L + \mathbf{i}_- \circ J_B(F)$ is also symplectic in H. Let $\tilde{F} = F^\perp \cap \lambda_+$, then the path $\{\boldsymbol{\gamma}(\mathbf{c}(t))\}$ is expressed as

$$\boldsymbol{\gamma}(\mathbf{c}(t)) = \{\cos(\pi t) \cdot x + \sin(\pi t) \cdot \mathbf{i}_- \circ J_B \circ \mathbf{i}_+(x) + z | x \in L, z \in (\mathbf{i}_+)^{-1}(\tilde{F})\}.$$

The path $\{\boldsymbol{\gamma}(\mathbf{c}(t))\}$ has only one non-trivial crossing at $t = 1/2$. We show that this is a regular crossing and determine the signature of the crossing form at $t = 1/2$.

Let $A = \mathbf{i}_- \circ J_B \circ \mathbf{i}_+$ and we take a suitable subspace $\tilde{K}$ in ℓ_- such that $\tilde{K} \cap A(L) = \{0\}$, $\tilde{K} + (\mathbf{i}_+)^{-1}(\tilde{F})$ is symplectic, and $L + \tilde{K} = \nu$ is a Lagrangian subspace transversal to $\{\boldsymbol{\gamma}(\mathbf{c}(1/2))\}$, so that we can express $\{\boldsymbol{\gamma}(\mathbf{c}(t))\}_{|t-1/2| \ll 1}$ as graphs of linear maps

$$f_t : \boldsymbol{\gamma}(\mathbf{c}(\tfrac{1}{2})) = A(L) + (\mathbf{i}_+)^{-1}(\tilde{F}) \to \nu, \qquad f_t : u + z \mapsto \cot(\pi t) \cdot A^{-1}(u).$$

Now we determine the crossing form at $t = 1/2$. Let $x, y \in L$, then

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\omega_H(A(x), f_t(A(y)))_{|t=1/2}\\
&\quad = \omega_H\left(\mathbf{i}_- \circ J_B \circ \mathbf{i}_+(x), \frac{-\pi}{\sin^2(\pi t)} y\right)_{|t=1/2}\\
&\quad = -\pi\omega_B(J_B \circ \mathbf{i}_+(x), \mathbf{i}_+(y)) = \pi\langle \mathbf{i}_+(x), \mathbf{i}_+(y)\rangle^B.
\end{aligned}$$

From this equality, both of the Maslov indexes $\mathbf{Mas}(\{\mathbf{c}(t)\}, \lambda_-) = 1$ and $\mathbf{Mas}(\{\boldsymbol{\gamma}(\mathbf{c}(t))\}, \ell_-) = 1$.

(F3) Let μ in $\mathcal{F}\Lambda_{\lambda_-}(B)$ and assume μ is contained in the Maslov cycle $\mathfrak{M}_{\lambda_-}(B)$ with $\dim \mu \cap \lambda_- = N > 0$.

Let $\{\mathbf{c}(t)\}$ be the path in $\mathcal{F}\Lambda_{\lambda_-}(B)$ defined in Example 3.31 in Section 3.4 for $F = \mathbf{i}_+ \circ J_H \circ \mathbf{i}_-(\mu \cap \lambda_-)$.

Note that here we used Corollary 5.7 for the existence of a compatible inner product in the symplectic Hilbert space B. So by the corresponding almost complex structure J_B, the isotropic subspaces $\mu\cap\lambda_-$ is written as $\mu\cap\lambda_- = J_B(\mathbf{i}_+\circ J_H\circ\mathbf{i}_-(\mu\cap\lambda_-))$. Then we can construct a path in Example 3.31 of Section 3.4 in terms of unitary operators.

Now again the path $\{\boldsymbol{\gamma}(\mathbf{c}(t))\}$ has only one non-trivial crossing at $t = 1/2$ with ℓ_- and by the same way as in (F2) we know that the crossing form is positive definite on $\mu\cap\lambda_-$.

(F4) Finally we can prove the coincidence of the Maslov indexes (5.2) for arbitrary continuous paths. Since if the given path $\{\mathbf{c}(t)\}$ is a loop, then they coincide because of the fact that they coincides for a generator of the fundamental group of the space $\mathcal{F}\Lambda_{\lambda_-}(B)$. If it is not a loop, then by joining the paths in (F3) from the end point which is in the Maslov cycle $\mathfrak{M}_{\lambda_-}(B)$ and we make this catenated path to a loop again by joining a path in (F1). Now we know the Maslov indexes of these loops coincides and Maslov indexes of added paths are all coincident, so that this prove Theorem 5.10(c).

6. Closed symmetric operators and Cauchy data spaces

In this section we discuss Lagrangian subspaces in the symplectic Hilbert space β explained in Example 2.2.

6.1. Cauchy data space

Let L be a real Hilbert space, and A be a closed densely defined symmetric operator with the domain $\mathfrak{D}_m \subset L$. We denote the domain of the adjoint operator A^* by $\mathfrak{D}_M$. As explained in the Example 2.2 the factor space $\boldsymbol{\beta} = \mathfrak{D}_M/\mathfrak{D}_m$ is a symplectic Hilbert space.

Even if we add a bounded selfadjoint operator B to the operator A, we have the same domain $\mathfrak{D}_M$ of the adjoint operator $(A+B)^*$, the graph norms defined on $\mathfrak{D}_M$ are equivalent and moreover the symplectic forms defined by the operator A and $A+B$ coincide.

In any case we denote by γ the natural projection map $\gamma : \mathfrak{D}_M \to \boldsymbol{\beta}$. It will be clear that $\gamma(\mathrm{Ker}\, A^*)$ is an isotropic subspace, but we need some assumptions on the operator A to show the closedness of it. We call the space $\gamma(\mathrm{Ker}\, A^*)$ "Cauchy data space".

Proposition 6.1. *Let $\mathfrak{D}$ be a subspace such that $\mathfrak{D}_m \subset \mathfrak{D} \subset \mathfrak{D}_M$. Then the restriction of the adjoint operator A^* to the domain $\mathfrak{D}$ is selfadjoint, if and only if, the factor space $\gamma(\mathfrak{D})$ is a Lagrangian subspace in $\boldsymbol{\beta}$.*

From now on we assume that:

[E1]: A has at least one selfadjoint "Fredholm" extension, that is, there exists a subspace $\mathfrak{D}$ (closed in the graph norm topology) such that $A_{\mathfrak{D}} = A^*_{|\mathfrak{D}}$ is selfadjoint and has the finite dimensional kernel and the image $A_{\mathfrak{D}}(\mathfrak{D})$ is closed in L and is of finite codimension.

[**E2**]: Ker $A^* \cap \mathfrak{D}_m = \{0\}$.

Remark 6.2. By the assumption [**E2**], $A^* : \mathfrak{D}_M \to L$ is surjective. For the case of differential operators, the condition [**E2**] is requiring that the operator satisfies the unique continuation property with respect to a boundary (or a hypersurface).

Both of these conditions [**E1**] and [**E2**] are satisfied by elliptic operators of Dirac type on compact manifolds. For such operators the unique continuation property holds with respect to any hypersurfaces. The space $\mathfrak{D}_m$ will be the minimal domain of the definition, i.e., the subspace of the first order Sobolev space with the vanishing boundary values, and the Cauchy data space will be realized in a subspace of distributions on the boundary manifold, that is, in the $-1/2$ order Sobolev space.

Proposition 6.3. *Under the assumptions* [**E1**] and [**E2**]

(a) $\gamma(\text{Ker } A^*)$ *is a Lagrangian subspace,*
(b) $\gamma(\text{Ker } A^*)$ *and* $\gamma(\mathfrak{D})$ *is a Fredholm pair*.

Proof. Since $A^*(\mathfrak{D})$ is a closed finite codimensional subspace, we have $(A^*)^{-1}(A^*(\mathfrak{D})) =$ Ker $A^* + \mathfrak{D}$ is a closed subspace in $\mathfrak{D}_M$ (equipped with the graph norm topology), hence $\gamma(\text{Ker } A^* + \mathfrak{D}) = \gamma(\text{Ker } A^*) + \gamma(\mathfrak{D})$ is closed in $\boldsymbol{\beta}$.

Again since Ker $A^* + \mathfrak{D}$ is closed and $\dim(\text{Ker } A^* \cap \mathfrak{D}) < +\infty$ we know that Ker $A^* + \mathfrak{D}_m$ is also close in $\mathfrak{D}_M$, and so $\gamma(\text{Ker } A^*) = \gamma(\text{Ker } A^* + \mathfrak{D}_m)$ must be close in $\boldsymbol{\beta}$, and that it is a closed isotropic subspace.

Now we have relations:

(a) $\dim \gamma(\text{Ker}^*) \cap \gamma(\mathfrak{D}) = \dim \text{Ker } A^* \cap \mathfrak{D}$,
(b) $\dim L/(\text{Ker } A^* + \mathfrak{D}) = \dim \text{Ker } A^* \cap \mathfrak{D}$.

So we have

$$\dim \gamma(\text{Ker } A^*) \cap \gamma(\mathfrak{D}) = \dim \gamma(\text{Ker } A^*)^\circ \cap \gamma(\mathfrak{D}) = \dim \boldsymbol{\beta}/(\gamma(\text{Ker } A^*) + \gamma(\mathfrak{D})) < \infty,$$

and hence

$$\gamma(\text{Ker } A^*) \cap \mathfrak{D} = \gamma(\text{Ker } A^*)^\circ \cap \mathfrak{D}, \qquad \gamma(\text{Ker } A^*) + \mathfrak{D} = \gamma(\text{Ker } A^*)^\circ + \mathfrak{D}.$$

From these equalities $\gamma(\text{Ker } A^*)$ is a Lagrangian subspace and $\gamma(\text{Ker } A^*) \in \mathcal{F}\Lambda_{\gamma(\mathfrak{D})}(\boldsymbol{\beta})$. □

Corollary 6.4. *Under the assumptions* [**E1**] and [**E2**] *the extension of A to* $\mathfrak{D}_m + \text{Ker } A^*$, $A^*_{|\mathfrak{D}_m + \text{Ker } A^*}$, *is a selfadjoint operator*.

Remark 6.5. The extension in the above Corollary 6.4 is called "Soft extension". This is also an interesting extension, although it is far from Fredholm operators. For example, in the paper [18] the asymptotic behavior of non-zero eigenvalues was investigated for a symmetric elliptic operator of even order on a bounded domain.

Example 6.6. Let

$$A = \mathbb{J}\left(\frac{\mathrm{d}}{\mathrm{d}t} + B\right) \tag{6.1}$$

be an ordinary differential operator acting on $C_0^\infty(0, 1) \otimes \mathbb{R}^{2N}$, where

$$\mathbb{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

and B is a $2N \times 2N$ symmetric matrix. In this case $\beta = \mathfrak{D}_M/\mathfrak{D}_m \cong \mathbb{R}^{2N} \boxplus \mathbb{R}^{2N}$ with the symplectic form $\mathbb{J} \boxplus -\mathbb{J}$. The cases treated in [12] reduce to this case (see also [13] and [26]).

Example 6.7. We describe an example of the Cauchy data space which can be realized in the distribution space on a manifold.

Let M be a manifold with boundary $\Sigma = \partial M$, and A be a first order symmetric elliptic operator acting on a space of smooth sections of a real smooth vector bundle E. Here we mean that the operator is symmetric, when it is symmetric on the space of smooth sections whose supports do not intersect with the boundary manifold Σ.

We assume that the unique continuation property holds for this operator A with respect to the boundary hypersurface Σ.

The minimal domain of the definition $\mathfrak{D}_m$ is the subspace of the first order Sobolev space consisting of sections with vanishing boundary values. Even for this case, it is not easy to determine the domain of the adjoint operator. It is a little bit bigger than the whole first order Sobolev space. The Cauchy data space β is included in the Sobolev space of order $-1/2$ on Σ ([19]).

Then we assume that A has a product form near the boundary hypersurface Σ in the following sense.

Let $\mathcal{N} \cong [0, 1] \times \Sigma$ is a neighborhood of Σ and on this neighborhood, the operator A takes the form

$$A = \sigma\left(\frac{\partial}{\partial t} + B\right),$$

where σ is a bundle automorphism of the restriction of E to $\mathcal{N}$, and is independent from the coordinate of the normal direction $t \in [0, 1]$. It is also skew-symmetric and satisfies $\sigma^2 = -\mathrm{Id}$. The operator B is selfadjoint, first order elliptic operator on the vector bundle $E_{|\Sigma}$, also independent from the normal variable t and satisfies the relation $\sigma \circ B + B \circ \sigma = 0$ by the symmetric assumption. Now we can characterize the Cauchy data space in the following form.

Let $\{e_k\}_{k\in\mathbb{Z}\setminus\{0\}}$, $e_k > 0$ for $k > N_0$, $e_k < 0$ for $k < -N_0$ and $e_k = 0$ for $|k| \leq N_0$, $k \neq 0$, be the eigenvalues of the boundary operator B and denote by $\{\phi_k\}_{k\in\mathbb{Z}\setminus\{0\}}$ the corresponding orthonormal eigensections. Then we define the spaces by

$$H^+ = \left\{\sum_{k<0 \text{ finite sum}} c_j\phi_j\right\}, \qquad H^- = \left\{\sum_{k>0 \text{ finite sum}} c_j\phi_j\right\}.$$

Let $\lambda_\pm$ be the closure of $H^\pm$ with respect to the $\pm 1/2$ order Sobolev norm respectively, then the direct sum $\lambda_+ + \lambda_- = \beta$ [2,19].

Then let $L_\pm$ be the closures of $H^\pm$ with respect to the L_2-norm, then we have two symplectic Hilbert spaces $L_2(M)$ and β satisfying the conditions [**CP1**], [**CP2**] and [**CP3**] in Section 5.2.

Remark 6.8. In the above Example 6.7, if the boundary of the manifold M is divided into two components Σ_0 and Σ_1, then the space of boundary values β is also divided into the sum $\beta = \beta_0 \oplus \beta_1$, where β_i is in the Sobolev space of order $-1/2$ on Σ_i ($i = 0, 1$). Now the Cauchy data space $\gamma(\mathrm{Ker}\, A^*)$ defines a closed symplectic transformation $S : \mathfrak{D} \to \beta_1$, where $\mathfrak{D} = \{x \in \beta_0 | \exists y \in \beta_1, (x, y) \in \gamma(\mathrm{Ker}\, A^*)\}$ and $S(x) = y, (x, y) \in \gamma(\mathrm{Ker}\, A^*)$. We should note that this follows from the unique continuation property, i.e., $(\beta_0 \oplus \{0\}) \cap \gamma(\mathrm{Ker}\, A^*) = \{0\}$, and $(\{0\} \oplus \beta_1) \cap \gamma(\mathrm{Ker}\, A^*) = \{0\}$. Also a selfadjoint Fredholm extension is given by the Atiyah–Patodi–Singer boundary condition for the case of the operators with product form near the boundary. Or more generally, even if it is not of a product form near the boundary, there are such extensions by global elliptic boundary conditions (see for example [28]).

6.2. Continuity of Cauchy data spaces

Let A be the symmetric operator as above satisfying the conditions [**E1**] and [**E2**]. Let $\{B_t\}_{t\in[0,1]}$ be a continuous family of bounded selfadjoint operators on the Hilbert space L. If each operator $A + B_t$ for $t \in [0, 1]$ satisfies the conditions [**E1**] "with a common domain" $\mathfrak{D}$, i.e., $(A+B_t)^*_{|\mathfrak{D}} = A_{\mathfrak{D}} + B_t$ ($t \in [0, 1]$) is selfadjoint and a Fredholm operator, and [**E2**], then we have a family of Lagrangian subspaces $\{\gamma(\mathrm{Ker}(A + B_t)^*)\}_{t\in[0,1]}$ in $\boldsymbol{\beta}$ and each of them and $\gamma(\mathfrak{D})$ is a Fredholm pair.

Proposition 6.9. *The family* $\{\gamma(\mathrm{Ker}(A + B_t)^*)\}_{t\in[0,1]}$ *is a continuous family. Hence it is a continuous path in* $\mathcal{F}\Lambda_{\gamma(\mathfrak{D})}(\boldsymbol{\beta})$.

Proof. It will be enough to prove at $t = 0$. So let $T_t : \mathfrak{D}_M \to L \oplus \mathrm{Ker}(A + B_0)^*$ be a map defined by $T_t(x) = (A + B_0)^*(x) \oplus \pi_0(x)$, where π_0 is a projection operator $\pi_0 : \mathfrak{D}_M \to \mathrm{Ker}(A + B_0)^*$. Then since T_0 is an isomorphism, for a sufficiently small $\epsilon > 0$ the maps T_t for $0 \le t \le \epsilon$ are also isomorphisms. Hence we have $\mathrm{Ker}(A + B_t)^* = (T_t)^{-1}(\{0\} \oplus \mathrm{Ker}(A + B_0)^*)$, and that the family $\{\mathrm{Ker}(A + B_t)^*\}_{0\le t\le\epsilon}$ is continuous at $t = 0$ since the family $\{(T_t)^{-1}\}_{0\le t\le\epsilon}$ is a continuous family. $\square$

Remark 6.10. If the operator $A^*_{|\mathfrak{D}} = A_{\mathfrak{D}}$ has a compact resolvent and a Fredholm operator, then for any selfadjoint bounded operator B the sum $A + B$ satisfies the condition [**E1**].

Remark 6.11. In the case of the paper [12], the family (= the family of operators of the form (6.1)) has varying domains $\{\mathfrak{D}\}$ where the operator is realized as a selfadjoint operator according to the each value of the parameter. But in this case the operator family can be transformed into the above case of a fixed domain of the definition as a selfadjoint realization by a continuous family of unitary operators [26]. It would not be clear whether we can do such transformations for the family of elliptic operators in the higher dimensions (Example 6.7).

6.3. Spectral flow and Maslov index

Finally we just formulate an equality between "Spectral flow" and "Maslov index" arising from the family of operators explained in the previous subsection.

Let $\mathcal{F}(L)$ be the space of bounded Fredholm operators defined on a Hilbert space L. It is a classifying space for the K-group. Then the non-trivial component of the subspace $\hat{\mathcal{F}}(L)$ consisting of selfadjoint Fredholm operators, we denote it by $\hat{\mathcal{F}}_*(L)$, is a classifying space for the K^{-1}-group (in the complex case) and KO^{-7}-group (in the real case). Both of their fundamental groups are isomorphic to $\mathbb{Z}$ [3]. These isomorphisms are given by an integer, so called, the spectral flow for a family of selfadjoint Fredholm operators [2]. This integer is also defined for continuous paths of selfadjoint Fredholm operators without any assumptions at the end points [27]. We do not state the definition, but is given in a similar way as the definition of the Maslov index we gave in this article, or rather it should be thought of the initiating method which was given in the paper [27] to define the spectral flow based on the basic spectral property of the Fredholmness of the operators.

Let L, A and $\{B_t\}_{t\in[0,1]}$ be as above, that is, the family $\{A+B_t\}$ acting on the Hilbert space L satisfies the conditions [**E1**] *with a common subspace $\mathfrak{D}$ on which the operators $(A+B_t)^*_{|\mathfrak{D}}$ are selfadjoint and Fredholm. Then we see that $(A+B_t+s)^*_{|\mathfrak{D}} = A_{\mathfrak{D}} + B_t + s$ is also a Fredholm operator for sufficiently small $|s| \ll 1$. Now instead of the condition* [**E2**] *we assume a stronger property*.

[**E2′**]: *There exists an $\epsilon > 0$ such that for each $t \in [0,1]$ and $|s| < \epsilon$*, $\mathrm{Ker}(A+B_t+s)^* \bigcap \mathfrak{D}_m = \{0\}$.

Remark 6.12. Of course this condition is satisfied by Dirac type operators.

Under these assumptions [**E1**] and [**E2′**], and with the common domain of the definition $\mathfrak{D}$ for the selfadjoint Fredholm realization, we have the following theorem.

Theorem 6.13. *The spectral flow for the family $\{A_{\mathfrak{D}} + B_t\}_{t\in[0,1]}$ and the Maslov index of the path of Cauchy data spaces $\{\gamma(\mathrm{Ker}(A+B_t)^*)\}_{t\in[0,1]}$ with respect to the Maslov cycle $\gamma(\mathfrak{D})$ coincides*.

We do not give a proof of this theorem. First it was proved in [12] that a coincidence between "Spectral flow" and "Maslov index of boundary data" for a family of ordinary differential operators (Example 6.6). In this case the family of ordinary differential operators arises as the family of the Euler equations of the symplectic action integral which is defined by two transversally intersecting Lagrangian submanifolds in a symplectic manifold and the Maslov index in this case is of the finite dimension (see also [26]). Then it was proved in [33] on three dimensional manifolds and generalized to higher dimensions in [25] for a family of Dirac operators $\{A_t\}_{t\in[0,1]}$ with invertible operators at the end points $t = 0, 1$. In these cases the Maslov indexes are that in the infinite dimension. We reproved the theorem in the above general form in [14]. There we also proved a general addition formula for the spectral flow when we decompose a manifold into two parts. To prove it we apply our reduction theorem in Section 4 of the Maslov index in the infinite dimensions. Such types of the formula were also investigated in several authors or believed to hold in a more general contexts [9,10,32]. In [14] we tried to make clear the meaning of the condition that the operators in the family are of the form, so called, product form near the separating boundary manifold (Example 6.7). This kind of restriction for the family will correspond to a condition (excision pair) assumed in the Mayer–Vietoris exact sequence of the singular homology theory.

Appendix A

In this appendix we gather up some of fundamental facts without proofs from the theory of functional analysis, on which our arguments heavily rely. Because the objects we will deal with are infinite dimensional spaces and their homotopical properties.

Our Hilbert spaces will be mostly real separable Hilbert spaces and the theorems we sum up here are valid for both real and complex cases if we do not state particularly. So let H be a separable Hilbert space with the inner product by $\langle \bullet, \bullet \rangle$ and as usual we denote the norm of the element $x \in H$ by $\|x\| = \sqrt{\langle x, x \rangle}$.

A.1. Topology of operator spaces

Theorem A.1 (Kuiper's Theorem). *Let H be an infinite dimensional real (complex or quaternionic) separable Hilbert space, then the group of linear isomorphisms, we denote it by $GL(H)$, is contractible. Note that the topology of $GL(H)$ is defined by the norm convergence and it is a topological group with this topology.*

Corollary A.2. *Let H be an infinite dimensional real (complex or quaternionic) separable Hilbert space, then the subgroup of $GL(H)$ consisting of linear isomorphisms which preserves the inner product is also contractible to a point. We will denote them by $\mathcal{O}(H)$ (orthogonal group) for the real case, $\mathcal{U}(H)$ (unitary group) for the complex case and $Sp(H)$ (symplectic group) for the quaternionic case.*

Theorem A.3 (Palais's Theorem). *Let B be a Banach space and we assume there is a sequence of projection operators $\{\pi_n\}_{n=1}^{\infty}$ onto finite dimensional subspaces $L_n = \pi_n(B)$ such that $L_n \subset L_{n+1}$ and for each $x \in B$, $\{\pi_n(x)\}$ converges to x in the sense of norm, that is, $\{\pi_n\}_{n=1}^{\infty}$ converges to the identity operator in the strong sense. Then for each open set $O \subset B$, the injection map $j : \text{ind-lim}_{\to} \pi_n(O) \to O$ is a homotopy equivalence.*

Let H be a real (or complex) Hilbert space, and by fixing a complete orthonormal basis $\{x_n\}_{n=1}^{\infty}$, we have inclusions of finite dimensional subspaces E_n, where E_n is spanned by $\{x_i\}_{i=1}^{n}$. Also from these inclusions of subspaces we have inclusions of the general linear groups $GL(n, \mathbb{R})$ (or $GL(n, \mathbb{C})$):

$$GL(n, \mathbb{R}) \subset GL(n + 1, \mathbb{R})$$

and

$$GL(n,\mathbb{C}) \subset GL(n+1,\mathbb{C})$$

in an obvious way. Then we have also inclusions $GL(n,\mathbb{R}) \to GL_K(H)$ (in the complex case $GL(n,\mathbb{C}) \to GL_K(H)$), where we denote

$$GL_K(H) = \{g \in GL(H) | g \text{ is of the form Id} + \text{compact operator}\}$$

corresponding to each case.

Proposition A.4. *The inclusion maps*

$$j : \text{ind-lim}_{\to} GL(n,\mathbb{R}) \to GL_K(H)$$

for the real case and

$$j : \text{ind-lim}_{\to} GL(n,\mathbb{C}) \to GL_K(H)$$

for the complex case, *are homotopy equivalences*.

Appendix B. Spectral notions

Let A be a densely defined closed operator (bounded or not bounded) on a Hilbert space H. Let $\lambda \in \mathbb{C}$, then λ is called a resolvent of the operator A, if $A-\lambda$ has a bounded inverse defined on the whole space H. We denote the set of all resolvents by $\rho(A)$. The complement $\mathbb{C} \setminus \rho(A)$ is called spectrum of A and we denote it by $\sigma(A)$. Let $\lambda \in \sigma(A)$, then if $A-\lambda$ has a densely defined inverse, but not continuous, then λ is called a continuous spectrum and we denote the subset consisting of continuous spectra by $\mathbf{C}_\sigma(A)$. Again let λ be in $\sigma(A)$ and assume that $A-\lambda$ is not invertible, that is $\{x \in H | (A-\lambda)(x) = 0\} \neq 0$, then such λ is called an eigenvalue or a point spectrum. We denote the set of eigenvalues by $\mathbf{P}_\sigma(A)$. The element in the complement in $\sigma(A)$ of the union $\mathbf{P}_\sigma(A) \cup \mathbf{C}_\sigma(A)$ is called residual spectrum, and we denote them by $\mathbf{R}_\sigma(A)$. Let $\lambda \in \mathbf{R}_\sigma(A)$, then $A-\lambda$ has an inverse, but the image $\mathrm{Im}(A-\lambda)$ is not dense.

Now let A be a selfadjoint operator (bounded or not bounded), then we know that there are no residual spectrum of A, that is, the spectrum $\sigma(A) = \mathbf{C}_\sigma(A) \cup \mathbf{P}_\sigma(A)$ and $\sigma(A) \subset \mathbb{R}$.

We denote by $\sigma_{\text{ess}}(A)$ a subset of $\sigma(A)$, each of which element is called "essential spectrum", if λ is an eigenvalue of infinite multiplicity or a continuous spectrum. If A is bounded selfadjoint, then $\sigma(A)$ is compact and $\|A\| = \sup_{t\in\sigma(A)} |t|$.

Let $\{E_t\}_{\{t\in\mathbb{R}\}}$ be a family of orthogonal projections defined on a Hilbert space H satisfying following properties (Sp 1), (Sp 2), (Sp 3) and (Sp 4), then we call $\{E_t\}_{\{t\in\mathbb{R}\}}$ a spectral measure:

$$(Sp\,1) \quad E_t(H) \subset E_s(H) \quad \text{for} \ \ t \le s,$$

$$(Sp\,2) \quad E_t \quad \text{is right strong continuous, that is, for each} \ \ x \in H,$$

$$\lim_{0<\delta\to 0} E_{t+\delta}(x) = E_t(x),$$

$$(Sp\,3)\quad \lim_{t\to\infty} E_t(x) = x \quad \text{for each} \ \ x \in H,$$

$$(Sp\,4)\quad \lim_{t\to-\infty} E_t(x) = 0 \quad \text{for each} \ \ x \in H.$$

Theorem B.1 (Spectral Decomposition Theorem). *Let A be a selfadjoint operator (bounded or not bounded) defined on a Hilbert space H. Then there is a unique spectral measure $\{E_t\}_{\{t\in\mathbb{R}\}}$ such that*

$$A = \int_{-\infty}^{\infty} t\,\mathrm{d}E_t.$$

Remark B.2. *The domain $\mathfrak{D}$ of the operator A is characterized as*

$$\mathfrak{D} = \left\{ x \in H | \int_{-\infty}^{+\infty} |t| d\|E_t(x)\|^2 < \infty \right\}.$$

Appendix C. Fredholm operators

Let H be a Hilbert space (or Banach space) and let T be a densely defined closed operator with the domain $\mathfrak{D}$. We call a closed operator T is a Fredholm operator, if it satisfies

$$\begin{array}{ll} \dim \operatorname{Ker}(T) & \text{is finite,} \\ \text{the image } \ \operatorname{Im}(T) = T(\mathfrak{D}) & \text{is closed,} \\ \dim \operatorname{Coker}(T) = H/\operatorname{Im}(T) & \text{is finite.} \end{array}$$

Remark C.1. For bounded Fredholm operators T we can prove that the image $T(H)$ is closed from the finite codimensionality of it by making use of the open mapping theorem.

Let $\mathcal{F}(H)$ be the space of all "*bounded*" Fredholm operators defined on a Hilbert space H.

Proposition C.2. *The space $\mathcal{F}(H)$ is an open subset in the space of all bounded operators $\mathcal{B}(H)$ with the topology of the norm convergence.*

Let $\mathcal{K}(H)$ be the two-sided ideal consisting of compact operators in $\mathcal{B}(H)$, then the quotient algebra $\mathcal{B}(H)/\mathcal{K}(H)$ is called Calkin algebra. If π denotes the natural projection map $\pi : \mathcal{B}(H) \to \mathcal{B}(H)/\mathcal{K}(H)$, then we have the following proposition.

Proposition C.3. $\pi^{-1}((\mathcal{B}(H)/\mathcal{K}(H))^*) = \mathcal{F}(H)$, *where $(\mathcal{B}(H)/\mathcal{K}(H))^*$ denotes the group consisting of the invertible elements in the Calkin algebra $\mathcal{B}(H)/\mathcal{K}(H)$.*

For a Fredholm operator (closed or bounded) T we denote by $\operatorname{ind}(T)$ the difference

$$\operatorname{ind}(T) = \dim \operatorname{Ker} T - \dim \operatorname{Coker} T,$$

and call it the "*Fredholm index*" of the operator. Especially for bounded Fredholm operators $T \in \mathcal{F}(H)$ we have the following theorem.

Theorem C.4.

$$\text{ind} : \mathcal{F}(H) \to \mathbb{Z}$$

is a locally constant function, in fact, it distinguishes the connected components (= *path-wise connected*) *of the space* $\mathcal{F}(H)$.

Remark C.5.

(a) If H is finite dimensional, then the quantity $\text{ind}(T)$ always vanishes. So this has an only meaning in the infinite dimension.
(b) In the paper [7] a similar result for the connected components is proved for the space of all closed Fredholm operators. The topology for such a space is introduced by embedding it into the space of bounded operators on the product space $H \times H$ as orthogonal projection operators onto the graphs.

Theorem C.6. *Let K be a compact operator on H and T be a bounded Fredholm operator, then* $T + K$ *is also a Fredholm operator and*

$$\text{ind}(T + K) = \text{ind}\, T.$$

Appendix D. Existence of a compatible symplectic structure

Proposition D.1. *Let* $(H, (\bullet, \bullet))$ *be a real Hilbert space and* ω *a bounded and non-degenerate skew-symmetric bilinear form on H. Then we can replace the inner product by another one* $\langle \bullet, \bullet \rangle$ *such that* $(H, \omega, \langle \bullet, \bullet \rangle, J)$ *is a compatible symplectic Hilbert space.*

Proof. Let A be the operator defined by

$$\omega(x, y) = (A(x), y).$$

Then A is bounded, skew-symmetric and invertible. Put $|A| = \sqrt{{}^t A \circ A}$, and the new inner product by $\langle x, y \rangle = (|A|(x), y)$. By this inner product we can express $\omega(x, y) = \langle J(x), y \rangle$, where $J = |A|^{-1} \circ A$. Now $J^2 = |A|^{-1} \circ A \circ |A|^{-1} \circ A = |A|^{-2} \circ A^2 = -\text{Id}$, and also $\langle J(x), J(y) \rangle = (|A| \circ |A|^{-1} \circ A(x), |A|^{-1} \circ A(y)) = (|A|^{-1\,t} A \circ A(x), y) = (|A|(x), y) = \langle x, y \rangle$. □

Corollary D.2. *Let H be a symplectic Hilbert space and we assume that H is polarized by two Lagrangian subspaces* λ *and* $\mu : H = \lambda \oplus \mu$. *Then there is an inner product with which the symplectic structure is compatible and the decomposition is orthogonal.*

Proof. In the above proof we can assume that the subspaces λ and μ are orthogonal with respect to the inner product $(\bullet, \bullet)$. Then the operator A, $(A(x), y) = \omega(x, y)$ maps λ to μ and

μ to λ. Hence ${}^t A \circ A$ and its square root keep the subspaces λ and μ invariantly, so that with the new inner product $(|A|(x), y)$ the Lagrangian subspaces λ and μ are again orthogonal. So the new inner product $(|A|(x), y)$ gives us the compatible symplectic structure. □

References

[1] V.I. Arnold, Characteristic class entering in quantization conditions, Funct. Anal. Appl. 1 (1967) 1–13.

[2] M.F. Atiyah, V.K. Patodi, I.M. Singer, Spectral asymmetry and Riemannian geometry: I, Math. Proc. Camb. Phil. Soc. 77 (1975) 43–69;
M.F. Atiyah, V.K. Patodi, I.M. Singer, Spectral asymmetry and Riemannian geometry: II, Math. Proc. Camb. Phil. Soc. 78 (1975) 405–432;
M.F. Atiyah, V.K. Patodi, I.M. Singer, Spectral asymmetry and Riemannian geometry: III, Math. Proc. Camb. Phil. Soc. 79 (1976) 71–99.

[3] M.F. Atiyah, I.M. Singer, Index theory for skew-adjoint Fredholm operators, Inst. Hautes Études Sci. Publ. Math. 37 (1969) 5–26.

[4] B. Booss-Bavnbek, K. Furutani, The Maslov index—a functional analytical definition and the spectral flow formula, Tokyo J. Math. 21 (1998) 1–34.

[5] B. Booss-Bavnbek, K. Furutani, Symplectic functional analysis and spectral invariants, in: B. Booss-Bavnbek, K.P. Wojciechowski (Eds.), Geometric Aspects of Partial Differential Equations, Am. Math. Soc. Series Contemporary Mathematics, vol. 242, Providence, RI, 1999, pp. 53–83.

[6] B. Booss-Bavnbek, K. Furutani, N. Otsuki, Criss-cross reduction of the Maslov index and a proof of the Yoshida–Nicolaescu Theorem, Tokyo J. Math. 24 (1) (2001).

[7] H.O. Cordes, J.P. Labrousse, The invariance of the index in the metric of closed operators, J. Math. Mech. 12 (1963) 693–719.

[8] S.E. Cappell, R. Lee, E.Y. Miller, On the Maslov index, Comm. Pure Appl. Math. 47 (1994) 121–186.

[9] S.E. Cappell, R. Lee, E.Y. Miller, Selfadjoint elliptic operators and manifold decompositions: Part I. Low eigenmodes and stretching, Comm. Pure Appl. Math. 49 (1996) 825–866;
S.E. Cappell, R. Lee, E.Y. Miller, Selfadjoint elliptic operators and manifold decompositions: Part II. Spectral flow and Maslov index, Comm. Pure Appl. Math. 49 (1996) 869–909;
S.E. Cappell, R. Lee, E.Y. Miller, Selfadjoint elliptic operators and manifold decompositions: Part III. Determinant line bundles and Lagrangian intersection, Comm. Pure Appl. Math. 52 (1999) 543–611.

[10] M. Daniel, P. Kirk, With an appendix by K.P. Wojciechowski: a general splitting formula for the spectral flow, Michigan Math. J. 46 (1999) 589–617.

[11] J.J. Duistermaat, On the Morse index in variational calculus, Adv. Math. 21 (1976) 173–195.

[12] A. Floer, A relative Morse index for the symplectic action, Comm. Pure Appl. Math. 41 (1988) 393–407.

[13] K. Furutani, N. Otsuki, Spectral flow and intersection numbers, J. Math. Kyoto Univ. 33 (1993) 261–283.

[14] K. Furutani, N. Otsuki, Maslov index in the infinite dimension and a splitting formula for a spectral flow, Jpn. J. Math. 28 (2) (2002) 215–243.

[15] K. Furutani, On the Quillen determinant, J. Geom. Phys. 49 (2004) 366–375.

[16] M. de Gosson, La définition de l'indice de Maslov sans hypothèse de transversalité, CR Acad. Sci. Paris, Série I 310 (1990) 279–282.

[17] M. de Gosson, The structure of q-symplectic geometry, J. Math. Pures Appl. 71 (1992) 429–453.

[18] G. Grubb, Spectral asymptotics for the soft selfadjoint extension of asymmetric elliptic differential operator, J. Oper. Thoery 10 (1983) 9–20.

[19] L. Hoermander, Pseudo-differential operators and non-elliptic boundary problems, Ann. Math. 83 (1966) 129–209.

[20] L. Hoermander, Fourier integral operators I, Acta Math. 127 (1–2) (1971) 79–183.

[21] L. Hoermander, The Analysis of Linear Partial Differential Operators III, Springer, Berlin, 1985.

[22] T. Kato, Perturbation Theory for Linear Operators, 2nd ed., Springer-Verlag, Berlin, 1980.

[23] N.J. Kalton, R.C. Swanson, A symplectic Banach space with no Lagrangian subspaces, Trans. Am. Math. Soc. 273 (1) (1982) 385–392.

[24] J. Leray, Analyse Lagrangiénne et mécanique quantique: Une structure mathématique apparentée aux développements asymptotiques et à l'indice de Maslov, Série Math. Pure et Appl., IRMP, Strasbourg, 1978 (English translation 1981, MIT Press).
[25] L. Nicolaescu, The Maslov index, the spectral flow, and decomposition of manifolds, Duke Math. J. 80 (1995) 485–533.
[26] N. Otsuki, K. Furutani, Spectral flow and Maslow index arising from Lagrangian intersections, Tokyo J. Math. 14 (1991) 135–150.
[27] J. Phillips, Self-adjoint Fredholm operators and spectral flow, Can. Math. Bull. 39 (1996) 460–467.
[28] J.V. Ralston, Deficiency indices of symmetric operators with elliptic boundary conditions, Comm. Pure Appl. Math. 23 (1970) 221–232.
[29] J. Robbin, D. Salamon, The Maslov index for paths, Topology 32 (1993) 827–844.
[30] J.M. Souriau, Construction explicite de l'indice de Maslov, in: Group Theoretical Methods in Physics, Springer Lecture Notes in Physics, vol. 50, Berlin, 1975, pp. 117–148.
[31] R.C. Swanson, Fredholm intersection theory and elliptic boundary deformation problems: I, J. Diff. Eqs. 28 (1978) 189–201.
[32] C.H. Taubes, Casson's invariant and gauge theory, J. Differ. Geom. 31 (1990) 547–599.
[33] T. Yoshida, Floer homology and splittings of manifolds, Ann. Math. 134 (1991) 277–323.